THETA CONSTANTS ASSOCIATED TO CUBIC THREE FOLDS

KEIJI MATSUMOTO AND TOMOHIDE TERASOMA

1. Introduction

Every elliptic curve is isomorphic to the double covering of the projective line $\mathbf{P}^1$ branching at four points. The set of the isomorphism classes of elliptic curves with marking of branching points can be identified with the configuration space $\mathcal{M}_{4pts}$ of ordered four points on $\mathbf{P}^1$. By considering their periods, one obtains a morphism *per* from $\mathcal{M}_{4pts}$ to the set $\mathcal{M}_{ct}$ of the isomorphism classes of complex torus with marking of 2-torsion points. The set $\mathcal{M}_{ct}$ can be identified with the quotient space $\mathfrak{H}/\Gamma(2)$ of the upper half plane $\mathfrak{H}$ by the principal congruence subgroup $\Gamma(2)$ of level 2 in $SL(2,\mathbf{Z})$. By the classical theory of elliptic functions, the map $per : \mathcal{M}_{4pts} \to \mathcal{M}_{ct}$ is an isomorphism. The inverse of *per* can be described in terms of celebrated Jacobi's theta constants.

Many mathematicians have tried to find moduli spaces of suitable algebraic varieties which can be uniformized by some symmetric space. E. Picard was the first person who found such moduli space which is two dimensional. He studied a family of Picard curves, which are cyclic triple coverings of $\mathbf{P}^1$ branching at five points. The periods of a curve determine an element of the 2-dimensional complex ball $\mathbf{B}_2$ embedded in the Siegel upper-half space $\mathfrak{H}_3$ of degree 3. This correspondence gives a uniformization of the moduli space of the Picard curves by $\mathbf{B}_2$. The inverse Θ of the period map was expressed in terms of special values of theta functions for the Jacobians of Picard curves. Shiga then found the representation of the map Θ in terms of theta constants, which are automorphic forms on $\mathbf{B}_2$ with respect to the monodromy group. Inspired by Picard's results, the moduli spaces of the cyclic coverings of $\mathbf{P}^1$ branching at $(n+3)$-points uniformized by the n-dimensional complex balls were classified by Terada ([T]), Deligne and Mostow ([DM]). One specific three dimensional moduli spaces listed in [DM] was studied in [Ma] similarly to Shiga: the inverse of the period map for a family of cyclic triple coverings of $\mathbf{P}^1$ branching at six points was expressed in terms of theta constants.

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Recently, Allcock, Carlson and Toledo showed that the moduli space of marked cubic surfaces can be uniformized by the 4-dimensional complex ball $\mathbf{B}_4$, though the period map for any family of cubic surfaces is constant. Let Y be the cyclic triple covering of $\mathbf{P}^3$ branching along a cubic surface X. The intermediate Jacobian $J(Y)$ of Y is a 5-dimensional abelian variety. By considering the normalized period matrix of $J(Y)$, they obtain a point τ in the Siegel upper half space $\mathfrak{H}_5$ of degree 5. Since the abelian variety $J(Y)$ admits an action of μ_3, τ belongs to the subdomain $\mathbf{B}_4 = \{\tau \in \mathfrak{H}_5 \mid (H\tau)^2 + H\tau + I = 0\}$ in $\mathfrak{H}_5$, where $H = \mathrm{diag}(1,1,1,1,-1)$ and μ_m is the group of m-th roots of unity. As a consequence, they get a multivalued holomorphic map φ from the moduli space $\mathcal{M}_{cs}$ of cubic surfaces with marking of the 27 lines to the 4 dimensional complex ball $\mathbf{B}_4$. Its image is an analytic Zariski open set of $\mathbf{B}_4$. In this manner, they get a period map. Moreover, the monodromy group of φ is an arithmetic subgroup Γ of the unitary group $U(4,1)$ and the induced holomorphic map $per : \mathcal{M}_{cs} \to \Gamma\backslash\mathbf{B}_4$ is a birational morphism. They define an action of $PO(5,\mathbf{F}_3) \simeq W(E_6)$ on $\Gamma\backslash\mathbf{B}_4$ which is compatible with the classical action of the Weyl group $W(E_6)$ on $\mathcal{M}_{cs}$ through the period map per.

In this paper, we study the action of $W(E_6)$ on the $(1-\rho)$-torsion subgroup $J(Y)_{1-\rho}(\simeq \mathbf{F}_3^5)$ of $J(Y)$, where ρ is an automorphism of Y of order 3. The intersection form on $H_2^{prim}(X,\mathbf{Z})$ defines a $\mathbf{F}_3$-valued quadratic form q invariant under the action of $W(E_6)$. To each element v in $J(Y)_{1-\rho}$, we assign a theta constant $\Theta_v(\tau)$ on $\mathbf{B}_4$. It is easy to see that $\Theta_v(\tau) \equiv 0$ for $v \notin S = \{v \in J(Y)_{1-\rho} \mid q(v) = 0\}$. The cubes of the non vanishing eighty $(= \#S)$ theta constants for $v \in S$ give a $W(E_6)$-equivariant projective embedding Θ of $\Gamma\backslash\mathbf{B}_4$. On the other hand, the moduli space of smooth cubic surfaces with marking of the 27 lines is isomorphic to the moduli space $\mathcal{M}_{6pts}$ of ordered 6 points on $\mathbf{P}^2$ in general position. After Coble, there exists a $W(E_6)$-equivariant projective embedding Z of $\mathcal{M}_{6pts}$ using 80 polynomials labeled by S. The main theorem (Theorem 5.7) of this paper asserts that the projective embeddings Θ and Z coincide via the period map φ. As a consequence, we obtain the inverse of the period map φ.

Here we explain the basic tool for our study. For a line L in X, we associate a μ_3-covering C of L branching at twelve points. This curve C admits an involution σ on C commuting with ρ and fixing exactly two points among the twelve, which are denote by $\{p_0 = \sigma(p_0), p_\infty = \sigma(p_\infty), p_1, \dots, p_5, \sigma(p_1), \dots, \sigma(p_5)\}$. The key fact in this paper is that the Prym variety $Prym(C,\sigma)$ is isomorphic to $J(Y)$. As a consequence, the periods of $J(Y)$ are equal to those of $Prym(C,\sigma)$. Note that the brancing index of the μ_6-covering $C \to C/<\sigma,\rho> \simeq \mathbf{P}^1$ appears in the list of Mostow's paper [Mo]. A proof of the surjectivity of the period map after Allcock, Carlson and Toledo reduces to Mostow's results. Via the isomorphism $Prym(C,\sigma) \simeq J(Y)$, the images of $p_1, \dots, p_5$ under the Abel-Jacobi map $C \to Prym(C,\sigma)$ form an orthonormal basis in $J(Y)_{1-\rho}$. Conversely any orthonormal basis of $J(Y)_{1-\rho}$ can be obtained from one of the 27 lines. Thus we have a one to one correspondence

between the 27 lines and orthonormal bases of $J(Y)_{1-\rho}$. This correspondence yields a dictionary between the geometry of the 27 lines and the geometry of $(\mathbf{F}_3^5, q)$. For example, two lines L_1 and L_2 intersect if and only if the corresponding orthonormal bases contain a common element up to signature. From this fact, we get a one to one correspondence between the 45 tritangents and the elements of length 1, modulo signature. Moreover we get a one to one correspondence between the 36 double sixes and the elements of length 2, modulo signature. As an application of this dictionary, we describe relations of degree 3 and 9 for theta constants in terms of the geometry of $(\mathbf{F}_3^5, q)$.

We would like to explain the contents of this paper. In Section 2, we study the line geometry on the cubic threefold Y obtained by the μ_3-covering of $\mathbf{P}^3$ branching along a cubic surface X. There we introduce a curve C, an involution σ and an automorphism ρ of order three. The main result in this section is Corollary 2.8: the Prym variety of C for the involution σ is isomorphic to the intermediate Jacobian $J(Y)$ of Y.

In Section 3, we introduce a symplectic basis $A_1, \cdots, A_5, B_1, \dots, B_5$ of the intermediate Jacobian $J(Y)$ of Y, which will be used to describe theta functions in Section 4. Thanks to the explicit description of the basis of topological cycles, we check that the images of the branch points under the Abel-Jacobi map form an orthonormal basis of the $(1-\rho)$-torsion subgroup $J(Y)_{1-\rho}$ of $J(Y)$. There we describe the correspondence between the 45 tritangents and the elements of length 1 in $\mathbf{F}_3^5$ and an explicit isomorphism between $PO(5, \mathbf{F}_3)$ and $Aut(\Gamma_{std})$, where Γ_{std} is the dual graph of a cubic surface.

We introduce the theta function associated to the Prym variety equipped with a symplectic basis. We study the pull-back of theta functions by morphisms $C \to Prym(C, \sigma)$.

In the last section, we state the main theorem. We define a projective embedding $Z : \mathcal{M}_{6pts} \to \mathbf{P}^{79}$ of $\mathcal{M}_{6pts}$ by 80 polynomials labeled by S. The advantage of using these polynomials is that they behave well under the action of $W(E_6)$. Using this description, we prove that the inverse of the period map can be described by theta constants defined in Section 4.

Before closing this introduction, we give some remarks on degenerations of cubic surfaces. A nodal cubic surface is obtained from the blowing up of six points on a conic C. The intermediate Jacobian of the μ_3-covering of the nodal cubic surface is isomorphic to the Jacobian of a μ_3-covering $\tilde{C}$ of the conic C branching at the six points. Actually, the genus of this curve is 4 and the image of the canonical map to $\mathbf{P}^3$ is a complete intersection of hypersurfaces H_2 and H_3 of degree 2 and 3 in $\mathbf{P}^3$. Let $\tilde{P}$ be the blowing up of $\mathbf{P}^3$ along the embedded curve $\tilde{C}$. Then we can contract the strict transform of H_2. The resulting threefold Y is nothing but the μ_3-covering of $\mathbf{P}^3$ branching along the nodal cubic surface. In the closure $\bar{\mathcal{M}}_{cs}$ of the image of $Z : \mathcal{M}_{cs} \to \mathbf{P}^{79}$, the component of the boundary $\bar{\mathcal{M}}_{cs} - \mathcal{M}_{cs}$ corresponds to the mirrors of the 36 reflections in $W(E_6)$. The detail will appear elsewhere.

Notations.

μ_m : The group of m-th roots of unity in $\mathbf{C}^\times$.
$\Im(\tau)$: The imaginary part of a complex matrix τ.
M_0 : The row vector consisting of the diagonal entries of an $n \times n$ matrix M.
$\mathbf{e}(x)$:= $\exp(2\pi\sqrt{-1}x)$.
$\mathbf{1}$:= $(1, \dots, 1)$.

2. Line geometry of cubic threefold with μ_3-actions

2.1. **A normal form for cubic surfaces.** In this section, we introduce a certain normal form for cubic surfaces. First we recall several well known facts about smooth cubic surfaces. Let X be a cubic surface, i.e. a smooth surface of degree 3 in $\mathbf{P}^3$. There are exactly 27 lines on the surface X and we can choose mutually disjoint 6 lines from them, which are written as $E_1, \dots, E_6$. Since the self-intersection numbers of these lines are -1, by contracting $E_1, \dots, E_6$ to points $P_1, \dots, P_6$, we obtain the two dimensional projective space $\mathbf{P}^2$. The set of points $P_1, \dots, P_6$ on $\mathbf{P}^2$ are generic in the following sense:

1. No three of $P_1, \dots, P_6$ are collinear. i.e. there exist no lines passing through three points among the six.
2. The six points $P_1, \dots, P_6$ are not coconic, i.e. there exist no conics passing through the six.

For $i = 1, \dots, 6$, there exists a unique conic $\bar{C}_i$ (resp. a line $\bar{L}_{ij}$) in $\mathbf{P}^2$ containing the P_j's $(j \neq i)$ (resp. P_i and P_j $(1 \le i < j \le 6)$). The proper transforms C_i and L_{ij} of $\bar{C}_i$ and $\bar{L}_{ij}$ are lines in X. Then $E_1, \dots, E_6, C_1, \dots, C_6$ and L_{ij} $(1 \le i < j \le 6)$ are the 27 distinct lines in the cubic surface X.

A notion of a marked cubic surface is defined as follows. We define the standard dual graph Γ_{std} as follows. (1) The set of the vertices consists of e_i, c_i $(i = 1, \dots, 6)$ and l_{ij} $(1 \le i < j \le 6)$. (2) e_i is adjacent to c_j if and only if $i \neq j$. (3) e_i (resp. c_i) is adjacent to l_{jk} if and only if $i \in \{j, k\}$. (4) l_{ij} is adjacent to l_{kl} if and only if $\{i, j\} \cap \{k, l\} = \emptyset$. (5) e_i and e_j (resp. c_i and c_j) $(i \neq j)$ does not adjacent to each other. Then the dual graph $\Gamma(X)$ of the 27 lines in X is isomorphic to Γ_{std}. A marking Ψ_{cs} of X is defined as an isomorphism $\Psi_{cs} : \Gamma(X) \to \Gamma_{std}$ of graphs.

Since P_6 is not contained in the conic $\bar{C}_6$ in $\mathbf{P}^2$, there are two tangent lines T_0 and T_∞ of $\bar{C}_6$ passing through P_6. The tangent points are denoted by Q_0 and Q_∞. If $\bar{L}_{i6}$ tangents to $\bar{C}_6$ in $\mathbf{P}^2$, then $L_{i6} \cap C_6 \cap E_i$ consists of one point P. This point P is called an Eckardt point of X. If a cubic surface is sufficiently generic, then there exist no Eckardt points. Until the end of this section, we assume that X has no Eckardt points. We choose a coordinate $(x_0 : x_1 : x_2)$ of $\mathbf{P}^2$ such that

1. the conic $\bar{C}_6$ is expressed as $x_0 x_1 = x_2^2$, and
2. the equations of the two tangent lines T_0 and T_∞ are $x_0 = 0$ and $x_1 = 0$, respectively.

Then P_i $(i=1,\dots,5)$ are given by $(\frac{1}{a_i}:a_i:1)$ $(a_i\neq 0)$ and P_6 by $(0:0:1)$. Now we define a polynomial $h(x)$ and s_i $(i=1,\dots,5)$ by

$$\begin{aligned} h(x)=&\prod_{i=1}^{5}(x-a_i)\\ =&x^5+s_1x^4+s_2x^3+s_3x^2+s_4x+s_5. \end{aligned}$$

The vector space of homogeneous polynomials of degree three on x_0,x_1,x_2 vanishing at 6 points $P_1,\dots,P_6$ is four dimensional. A basis of this space is given by

$$\begin{aligned} u_0&=(x_0x_1-x_2^2)x_0,\\ u_1&=(x_0x_1-x_2^2)x_1,\\ u_2&=x_1^3+s_1x_1^2x_2+s_2x_1x_2^2+s_3x_0x_1x_2+s_4x_0x_2^2+s_5x_0^2x_2,\\ u_3&=x_1^2x_2+s_1x_1x_2^2+s_2x_0x_1x_2+s_3x_0x_2^2+s_4x_0^2x_2+s_5x_0^3. \end{aligned}$$

By eliminating x_0,x_1,x_2, we get the cubic relation $F(u_0,u_1,u_2,u_3)=0$, where

$$\begin{aligned} F(u_0,u_1,u_2,u_3)=&u_0u_2^2+(s_2u_0u_1+s_4u_0^2-u_1^2)u_2\\ &-(u_1u_3^2+(s_3u_0u_1+s_1u_1^2-s_5u_0^2)u_3)\\ &-(s_2u_1^3+s_4u_1^2u_0-s_1s_5u_0^2u_1-s_3s_5u_0^3). \end{aligned}$$

This is the defining equation of the cubic surface X. Under these coordinates $(u_0:\dots:u_3)$, the proper transform C_6 of the conic $\bar{C}_6$ is given by $u_0=u_1=0$. Any point of $\bar{C}_6$ different from Q_0 and Q_∞ is expressed as $(\frac{1}{x}:x:1)$, $(x\neq 0)$ and the corresponding point in the proper transform C_6 is given by $u_0=u_1=0$, $u_2=\frac{1}{x^2}h(x)$ and $u_3=\frac{1}{x^3}h(x)$. The projective coordinates of this point are given as

$$(u_0:u_1:u_2:u_3)=(0:0:x:1).$$

2.2. **Triple covering of $\mathbf{P}^3$ branching along the cubic surface.** Let $F=F(u_0,u_1,u_2,u_3)$ be the defining equation of the cubic surface X as in §2.1. The cubic threefold Y in $\mathbf{P}^4$ defined by $u_4^3=F(u_0,u_1,u_2,u_3)$ is a cyclic covering of $\mathbf{P}^3$ branching along $X\simeq\{(u_0:\dots:u_4)\in Y \mid u_4=0\}$. We define an action ρ of $\zeta\in\mu_3$ on Y by

$$\rho:(u_0:\dots:u_3:u_4)\mapsto(u_0:\dots:u_3:\zeta u_4).$$

Definition 2.1. *The subvariety of the Grassmann variety $Gr(5,2)$ of lines in $\mathbf{P}^4$ consisting of lines in Y is called the Fano variety $F(Y)$ of Y. (see* [CG]*.) For a line l in Y, the subvariety $Inc(l)$ of $F(Y)$ consisting of lines which intersect with l is called the incidental subvariety of l.*

Remark 2.2. *It is known (c.f.*[CG]*) that the Fano variety $F(Y)$ is a smooth surface and that $Inc(l)$ is a divisor of $F(Y)$.*

Since X is the branch locus of the covering $Y \to \mathbf{P}^3$, a line L in X is a line in Y. In the rest of this section, we investigate the incidental subvariety $Inc(C_6)$ of the line C_6 in Y and its normalization C.

Let P be a point of C_6 in Y different from Q_∞, Q_0. Then its coordinates are $(u_0 : u_1 : u_2 : u_3 : u_4) = (0 : 0 : x : 1 : 0)$ $(x \neq 0)$. The line L joining two points $(0 : 0 : x : 1 : 0)$ and $(a : b : c : 0 : e)$ can be parameterized as $(u_0 : \cdots : u_4) = (at, bt, ct+x, 1, et)$, $(t \in \mathbf{C} \cup \{\infty\})$. We consider the condition for a, b, c, e so that the line L is contained in Y, i.e. $F(at, bt, ct+x, 1) = (et)^3$ holds for all t. By the straight forward calculation, we have

$$\begin{aligned} ax^2 =& b, \\ 2acx + (s_2ab + s_4a^2 - b^2)x =& s_3ab + s_1b^2 - s_5a^2, \\ ac^2 + (s_2ab + s_4a^2 - b^2)c =& (s_2b^3 + s_4b^2a - s_1s_5a^2b - s_3s_5a^3) + e^3. \end{aligned}$$

By putting $a = 1$, we get $b = x^2$ and

$$\begin{aligned} c =& \frac{1}{2x}(x^5 + s_1x^4 - s_2x^3 + s_3x^2 - s_4x - s_5), \\ 4x^2e^3 =& h(x)h(-x). \end{aligned}$$

By setting $y = 4^{1/3}xe$, we get a family of lines contained in Y parameterized by the curve C^0:

$$y^3 = xh(x)h(-x) = x\prod_{i=1}^{5}(a_i^2 - x^2) \quad (x \neq 0, \infty). \tag{2.1}$$

By the map $C \ni (y, x) \to (\frac{1}{x} : x : 1) \in C_6$, C is regarded as a μ_3-covering of C_6. By direct computation, we can readily show that there are no lines passing through Q_0 (resp. Q_∞) other than C_6. Therefore $Inc(C_6) = C^0 \cup \{[C_6]\}$, where $[C_6]$ is the point of $Gr(5, 2)$ corresponding to the line C_6. Since $Inc(C_6)$ has two tangents at $[C_6]$ in $Gr(5, 2)$, it is a curve only with a node. Thus we get a family of lines in Y parameterized by C.

Note that the action of μ_3 on the Fano variety is compatible with the action of $\zeta \in \mu_3$ on the curve C defined by $(y \to \zeta y, x \to x) \in Aut(C)$. The action of $\omega = \dfrac{-1+\sqrt{-3}}{2}$ is denoted by ρ.

2.3. **Cylinder map.** In this section, we introduce and investigate the cylinder map induced by the family of lines defined in §2.2. Let $\mathcal{U}$ be the universal family defined by $\mathcal{U} = \{(p, x) \mid p \in C, x \in l_p\}$, where l_p is the line corresponding to the point $p \in C$, and let $pr_1 : \mathcal{U} \to C$ and $pr_2 : \mathcal{U} \to Y$ be the natural projections. Then pr_1 induces an isomorphism:

$$pr_1^* : H^1(C, \mathbf{Z}(-1)) \xrightarrow{\simeq} H^1(\mathcal{U}, \mathbf{Z}(-1)).$$

The Gysin map pr_{2*} is the Poincare dual of the natural homomorphism pr_2^* : $H^3(Y, \mathbf{Z}(3)) \to H^3(\mathcal{U}, \mathbf{Z}(3))$, i.e. $pr_{2*}b$ $(b \in H^1(\mathcal{U}, \mathbf{Z}(-1)))$ is an element so

that

$$a \cup pr_{2*}b = pr_2^* a \cup b \tag{2.2}$$

holds for any $a \in H^3(Y, \mathbf{Z})$. We define the cylinder map:

$$c : H^1(C, \mathbf{Z}(-1)) \xrightarrow{pr_1^*} H^1(\mathcal{U}, \mathbf{Z}(-1)) \xrightarrow{pr_{2*}} H^3(Y, \mathbf{Z}).$$

This is a homomorphism of Hodge structures.

We define an involution σ on C by $y \to -y$ and $x \to -x$.

Definition 2.3. *An involution σ' on C is called the Clemens-Griffiths involution if l_p and $l_{\sigma'(p)}$ are contained in a plane in $\mathbf{P}^4$ for generic $p \in C$.*

Proposition 2.4. *The involution σ coincides with the Clemens-Griffiths involution. Moreover the action of σ commutes with the action of ρ. As a consequence the group μ_6 of 6-th roots of unity acts on the curve C.*

Proof. By the definition of σ, if l_p is a line connecting $(0 : 0 : x : 1 : 0)$ and $(1 : b : c : 0 : e)$, then $l_{\sigma(p)}$ is a line connecting $(0 : 0 : -x : 1 : 0)$ and $(1 : b : c' : 0 : e)$. Therefore both l_p and $l_{\sigma(p)}$ are contained in the plane $bu_0 = u_1, eu_0 = u_4$. The commutativity of the action of ρ and σ is a direct consequence of the definition of σ and ρ. □

Corollary 2.5. *Let $H^1(C, \mathbf{Z}(-1))^+ = \{v \in H^1(C, \mathbf{Z}(-1)) \mid \sigma^*(v) = v\}$, and $H^1(C, \mathbf{Z}(-1))^- = H^1(C, \mathbf{Z}(-1))/H^1(C, \mathbf{Z}(-1))^+$. Then the cylinder map c factors through $H^1(C, \mathbf{Z}(-1))^-$.*

Proof. By Proposition 2.4, $H^1(C, \mathbf{Z}(-1))^-$ is the maximal quotient on which the Clemens-Griffiths involution acts as the (-1)-multiplication. On the other hand, the Clemens-Griffiths involution acts on $H^3(Y, \mathbf{Z})$ as the (-1)-multiplication and we obtain the corollary. □

The induced map $H^1(C, \mathbf{Z}(-1))^- \to H^3(Y, \mathbf{Z})$ is denoted by ϕ. The algebra over $\mathbf{Z}$ generated by ρ with the relation $1 + \rho + \rho^2 = 0$ is denoted by $\mathbf{Z}[\rho]$. Then $H^1(C, \mathbf{Z}(-1))^-$ and $H^3(Y, \mathbf{Z})$ are modules over $\mathbf{Z}[\rho]$.

Theorem 2.6. *The morphism ϕ is an isomorphism as $\mathbf{Z}[\rho]$ modules.*

Proof. Since the action of ρ is compatible, ϕ is a homomorphism as $\mathbf{Z}[\rho]$ modules. We will prove that this is actually an isomorphism. Let D be the quotient of C by the involution σ. Then by the Hurwitz theorem, the genus $g(C)$ and $g(D)$ of C and D are 10 and 5, respectively. Since the rank of $H^3(Y, \mathbf{Z})$ is 10, it is enough to prove the surjectivity.

We use the same notations C_6, $Inc(C_6) \subset Gr(5, 2)$ as in the last paragraph. Let $g : \mathcal{Y} \to \Delta$ be a small deformation of the cubic threefold $\mathcal{Y}_0 = Y$ over $\Delta = \{t \in \mathbf{C} \mid\mid t \mid < \epsilon\}$ such that the generic fibers $\mathcal{Y}_t$ at $t \in \Delta^* = \Delta - \{0\}$ are sufficiently generic. We can extend a line C_6 to a family of lines $\mathcal{C} \to \Delta$ contained in $\mathcal{Y}$. The family of incidental subvarieties $f : \mathcal{D} \to \Delta$ is a family of curves on Δ. By [CG], the generic fiber $\mathcal{D}_t$ of $\mathcal{D}$ at $t \in \Delta^*$ is a smooth curve of genus 11. Moreover the family of the Clemens-Griffiths involutions on the

fibers comes to be an involution σ of $\mathcal{D}$ preserving each fiber. Thus we have the relative cylinder map

$$\mathbf{R}^1 f_* \mathbf{Z}(-1) \to \mathbf{R}^3 g_* \mathbf{Z},$$

and it factors through the maximal quotient $(\mathbf{R}^1 f_* \mathbf{Z}(-1))^-$ of $\mathbf{R}^1 f_* \mathbf{Z}(-1)$ on which the involution σ acts as the (-1)-multiplication:

$$(\mathbf{R}^1 f_* \mathbf{Z}(-1))^- \to \mathbf{R}^3 g_* \mathbf{Z}.$$

Since g is a proper smooth morphism of cubic threefolds, $\mathbf{R}^3 g_* \mathbf{Z}$ is a smooth $\mathbf{Z}$-sheaf. By considering the rank of fibers and the specialization map, we get the smoothness of the sheaf $(\mathbf{R}^1 f_* \mathbf{Z}(-1))^-$. On the other hand, by Theorem 11.19 of [CG], for $t \in \Delta^*$ the homomorphism

$$(\mathbf{R}^1 f_* \mathbf{Z}(-1))^-_t \to \mathbf{R}^3 g_* \mathbf{Z}_t$$

is surjective. Therefore

$$(\mathbf{R}^1 f_* \mathbf{Z}(-1))^-_0 \to \mathbf{R}^3 g_* \mathbf{Z}_0$$

is surjective. Since $Inc(C_6)$ is a curve with one node, whose normalization C is a curve of genus 10, we have

$$(\mathbf{R}^1 f_* \mathbf{Z}(-1))^-_0 \simeq H^1(C, \mathbf{Z})^-,$$

and we get the theorem. □

Remark 2.7. *By the compatibility of cup products (2.2), the polarization of $J(Y)$ given by the cup product is equal to the half of the restriction of the cup product on $H^1(C, \mathbf{Z})$ to $H^1(C, \mathbf{Z})^-$.*

Let C, $\sigma : C \to C$ and $D = C/ < \sigma >$ be as in the proof of Theorem 2.6. Since the double covering $C \to D$ branches at two points, the Prym variety $Prym(C, \sigma) = Ker(J(C) \to J(D))$ is a principally polarized abelian variety (c.f.[Mu]). This is isomorphic to the image of the morphism

$$(1 - \sigma) : J(C) \to J(C) : v \to v - \sigma(v). \tag{2.3}$$

Via this isomorphism, $Prym(C, \sigma)$ is regarded as the maximal quotient of $J(C)$ on which the involution σ acts as (-1)-multiplication. We have the following corollary to Theorem 2.6.

Corollary 2.8. *Let $cor_C : J(C) \to J(Y)$ be the homomorphism induced by the correspondence $C \leftarrow \mathcal{U} \to Y$, where $J(Y)$ is the intermediate Jacobian of Y. The map $\bar{c} : Prym(C, \sigma) \simeq J(Y)$ induced by $-cor_C$ is an isomorphism.*

By the commutativity of σ and ρ, ρ induces an automorphism of $Prym(C, \sigma)$. The induced automorphism is also denoted by ρ. It is easy to see that $\bar{c}$ is compatible with the action of $\mu_3 =< \rho >$.

2.4. **The Abel-Jacobi map and the level map Λ.** We consider the following commutative diagram:

$$\begin{tikzcd}
H_2^{prim}(X,\mathbf{Z}) \arrow[r] \arrow[dd, "\lambda"] & H_2(X,\mathbf{Z}) \arrow[r] & H_2(\mathbf{P}^3,\mathbf{Z}) \\
 & CH_1(X) \arrow[u, "\simeq"'] \arrow[r] \arrow[d] & CH_1(\mathbf{P}^3) \arrow[u, "\simeq"'] \arrow[d] \\
A_1(Y) \arrow[r] \arrow[d, "a"] & CH_1(Y) \arrow[r] & CH_1(\mathbf{P}^4) \\
J(Y) & &
\end{tikzcd}$$

where $CH_1(X)$ is the Chow group of X of dimension 1, $A_1(Y)$ is the subgroup of $CH_1(Y)$ consisting of algebraic cycles algebraically equivalent to zero, and $H_2^{prim}(X,\mathbf{Z})$ is the kernel of the natural homomorphism $H_2(X,\mathbf{Z}) \to H_2(\mathbf{P}^3,\mathbf{Z})$. Let l and l' be lines in X disjoint. Then the images of lines l and l' in Y are algebraically equivalent. Thus the image of $[l]-[l']$ in $CH_1(Y)$ is contained in $A_1(Y)$. Since $H_2^{prim}(X,\mathbf{Z})$ is generated by elements $[l]-[l']$, with $l \cap l' = \emptyset$, the image of $H_2^{prim}(X,\mathbf{Z})$ is contained in $A_1(Y)$. The induced map $H_2^{prim}(X,\mathbf{Z}) \to A_1(Y)$ is denoted by λ. By composing λ and the Abel-Jacobi map $a : A_1(Y) \to J(Y)$, we get a homomorphism $\Lambda = a \circ \lambda : H_2^{prim}(X,\mathbf{Z}) \to J(Y)$, which is called the level map for X.

Let X be a cubic surface with a marking Ψ_{cs}. We define an element v_i in $J(Y)$ by

$$v_i = \Lambda([E_i]-[L_{i6}]).$$

The point of C defined by $x = a_i, y = 0$ (resp. $x = 0, \infty$) in the equation (2.1) is denoted as p_i (resp. p_0, p_∞). The point p_i (resp. $\sigma(p_i)$) corresponds to the line E_i (resp. L_{i6}). We define two morphisms $jac : C \to J(C)$ and $\jmath : C \to Prym(C,\sigma)$:

$$\begin{aligned}
&C \ni p \mapsto [p]-[p_0] \in J(C), \\
&C \ni p \mapsto [p]-[\sigma(p)] \in Prym(C,\sigma),
\end{aligned}$$

where $[p]$ is the divisor class of p. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
 & C & \xrightarrow{\jmath} & Prym(C) \\
(2.4) & jac \downarrow & (1-\sigma) \nearrow & \downarrow \bar{c} \\
 & J(C) & \underset{-cor_C}{\longrightarrow} & J(Y).
\end{array}$$

Proposition 2.9. *Under the composite map $\bar{c} \circ \jmath$*

$$C \xrightarrow{\jmath} Prym(C,\sigma) \overset{\bar{c}}{\simeq} J(Y),$$

the image of $p_i \in C$ is v_i.

Proof. By the definition of jac and $\bar{c}$, we have

$$jac(p) = [p] - [p_0], \quad -cor([p] - [p_0]) = -\Lambda([E_i] - [C_6]).$$

Since the involution σ induces the (-1)-multiplication on $Prym(C, \sigma)$, we have $-\Lambda([E_i] - [C_6]) = \Lambda([E_i] - [L_{i6}])$. The diagram (2.4) yields the proposition. □

3. Finite geometry for $(1-\rho)$-torsion subgroup of the Prym variety

3.1. **Symplectic basis for the curve C and $Prym(C, \sigma)$.** In this section we introduce a symplectic basis of $J(C)$ and $Prym(C, \sigma)$ compatible with the action of ρ. We assume that the cubic surface X has no Eckardt points in this section. To specify topological cycles of C, we assume that $a_1, \dots, a_5 \in \mathbf{R}$ and $0 < a_1 < \cdots < a_5$. The curve C can be expressed as μ_6 covering of $\mathbf{P}^1 \simeq C/ < \rho, \sigma >$: put $\xi = x^2$ in (2.1), then the curve C is given by

$$y^6 = \xi \prod_{i=1}^{5} (\xi - a_i^2)^2. \tag{3.1}$$

Note that $x = \prod_{i=1}^{5} (a_i^2 - \xi)^{-1} y^3$. The actions of ρ and σ are given by

$$\rho(y) = \omega y, \quad \rho(\xi) = \xi,$$
$$\sigma(y) = -y, \quad \sigma(\xi) = \xi.$$

By gluing 6 copies of the ξ-planes cut along the slit as in Figure 1, we get the curve C. Here the projection of the point p_i to ξ plane is denoted by ξ_i for short. We define topological cycles $\beta_1, \dots, \beta_5$ as in the Figure 1. In Figure 1 the numbers written along paths are the label of sheet through which the paths are passing. Here the sheets are labeled with $\mathbf{Z}/6\mathbf{Z}$. The action of σ (resp. ρ) sends the i-th sheet to the $(i+3)$-th sheet (resp. $(i+2)$-th sheet).

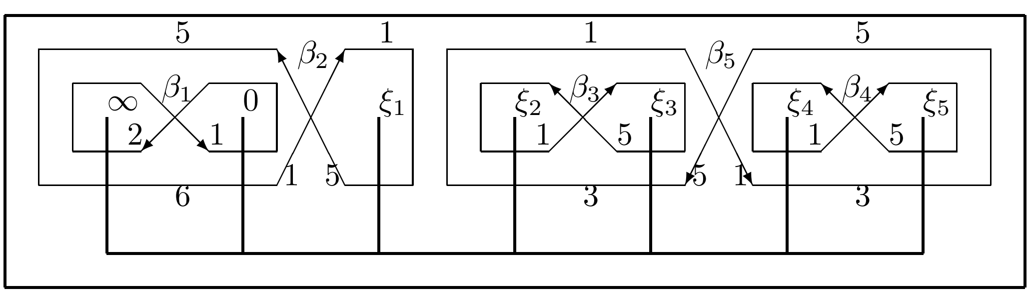

Figure 1. $\beta_1, \dots, \beta_5$.

Put

$$\begin{cases} \alpha_1 &= -\sigma_* \rho_* (\beta_1), \\ \alpha_i &= \rho_* \beta_i \quad (\text{for} \quad i = 2, 3, 4), \\ \alpha_5 &= -\rho_* \beta_5, \end{cases}$$

and $\alpha_i' = \sigma\alpha_i$ and $\beta_i' = \sigma\beta_i$ for $i = 1, \dots, 5$. Then we have

$$\alpha_i \cdot \alpha_j = 0, \quad \beta_i \cdot \beta_j = 0, \quad \alpha_i \cdot \beta_j = -\delta_{ij}.$$

Therefore the 1-cycles $\alpha_1, \dots, \alpha_5, \alpha_1', \dots, \alpha_5'$ and $\beta_1, \dots, \beta_5, \beta_1', \dots, \beta_5'$ form a symplectic basis for the cup product on $H_1(C, \mathbf{Z})$.

The inclusion $Prym(C, \sigma) \to J(C)$ corresponds to the inclusion $H_1(C, \mathbf{Z})^- \to H_1(C, \mathbf{Z})$, where $H_1(C, \mathbf{Z})^-$ is the (-1)-part of the action σ_*. We have the following proposition.

Proposition 3.1. *The restriction of the half of the cup product on $H_1(C, \mathbf{Z})$ to $H_1(C, \mathbf{Z})^-$ gives a principal polarization on $Prym(C, \sigma)$; that is*

$$A_i \cdot A_j = 0, \quad B_i \cdot B_j = 0, \quad A_i \cdot B_j = -2\delta_{ij},$$

where

$$A_i = \alpha_i - \alpha_i', \quad B_i = \beta_i - \beta_i'.$$

$H_1(C, \mathbf{Z})^-$ is a free $\mathbf{Z}[\rho]$-module of rank 5 generated by $B_1, \dots, B_5$.

Remark 3.2. *By the equation (3.1) of C, C can be regarded as a μ_6 covering of $\mathbf{P}^1$ branching at $0, \infty, a_1^2, \dots, a_5^2$. The branching index of this covering is $(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ under the notation of* [DM],[Mo]. *This index is contained in the table of* [Mo]. *This fact gives another proof of the surjectivity of the period map. (See §5.1 and Theorem 5.7.) Moreover, using the coordinate $a_1^2, \dots, a_5^2$, the period map of $J(Y)$ can be expressed by Appell's hypergeometric functions.*

3.2. **Orthonormal basis and the 27 lines.** As in §3.1, we assume that $0 < a_1 < \cdots < a_5$, and regard C as a 6-ple covering of the ξ-plane. Let γ_0, γ_1 and γ_i $(i = 2, \dots 5)$ be the branches of paths connecting ∞ and 0, 0 and ξ_1, and ξ_{i-1} and ξ_i $(i = 2, \dots, 5)$, respectively, illustrated in Figure 2.

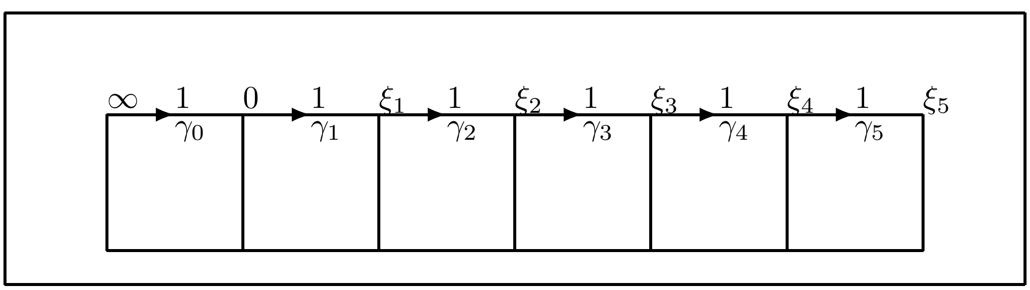

FIGURE 2. $\gamma_0, \dots, \gamma_5$

An $\mathbf{R}$ linear combination of homology classes of a closed paths defines an element of the $\mathbf{C}$-dual $H^0(C, \Omega^1)^*$ of $H^0(C, \Omega^1)$ by integration and this correspondence defines an isomorphism $H_1(C, \mathbf{R}) \to H^0(C, \Omega^1)^*$. Since this isomorphism is equivariant under the action of σ, the (-1)-eigen space $H_1(C, \mathbf{R})^-$ for σ is isomorphic to the $\mathbf{C}$-dual of $H^0(C, \Omega^1)^-$. A path not necessary closed

also defines an element of $H^0(C,\Omega^1)^*$. From now on, a path means the corresponding element in $H_1(C,\mathbf{R})$. The class of a path from p to q on C in $H_1(C,\mathbf{R})/H_1(C,\mathbf{Z})(\simeq J(C))$ corresponds to the divisor class $[q]-[p]\in J(C)$ by Abel's theorem.

The paths $\beta_1,\dots,\beta_5$ can be expressed as

$$\begin{aligned}
\beta_1 &= \gamma_0-\rho^2\sigma\gamma_0,\\
\beta_2 &= -\rho^2\gamma_0-\rho^2\gamma_1+\gamma_1+\rho\sigma\gamma_0,\\
\beta_3 &= -\rho^2\gamma_3+\gamma_3,\\
\beta_4 &= -\rho^2\gamma_5+\gamma_5,\\
\beta_5 &= \gamma_3+\gamma_4+\rho\gamma_5-\rho^2\gamma_5-\rho^2\gamma_4-\rho\gamma_3.
\end{aligned}$$

By the above equality, we have $(1-\sigma)\gamma_i\in\frac{1}{1-\rho}H_1(C,\mathbf{Z})^-=\{v\in H_1(C,\mathbf{Q})^-\mid (1-\rho)v\in H_1(C,\mathbf{Z})^-\}$. We put $\bar\gamma_i=(1-\sigma)\gamma_i \pmod{H_1(C,\mathbf{Z})^-}$. By the equality

$$\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5-\rho^2\gamma_5-\rho\gamma_4-\gamma_3-\rho^2\gamma_2-\rho\gamma_1-\rho^2\sigma\gamma_0=0,$$

we have

$$\bar\gamma_1=\bar\delta_1+\bar\delta_2,\quad \bar\gamma_2=-2\bar\delta_2+\bar\delta_3-2\bar\delta_5,\quad \bar\gamma_3=\bar\delta_3,$$
$$\bar\gamma_4=-2\bar\delta_3+\bar\delta_4+\bar\delta_5,\quad \bar\gamma_5=\bar\delta_4,$$

where $\bar\delta_i$ is the class of $\frac{1}{1-\rho^2}B_i$ modulo $H_1(C,\mathbf{Z})^-$. Since $\sum_{j=1}^i\gamma_j$ is a path from p_0 to p_i, $(1-\sigma)\sum_{j=1}^i\gamma_j$ is a path from $\sigma(p_i)$ to p_i, which corresponds to the divisor class $[p_i]-[\sigma(p_i)]$. By Proposition prop:cor vi and ai, we have $v_i=\sum_{j=1}^i\bar\gamma_j$ for $i=1,\dots,5$. More explicitly, we have

$$v_1=\bar\delta_1+\bar\delta_2,\quad v_2=\bar\delta_1-\bar\delta_2+\bar\delta_3+\bar\delta_5, \tag{3.2}$$

$$v_3=\bar\delta_1-\bar\delta_2-\bar\delta_3+\bar\delta_5,\quad v_4=\bar\delta_1-\bar\delta_2+\bar\delta_4-\bar\delta_5,\quad v_5=\bar\delta_1-\bar\delta_2-\bar\delta_4-\bar\delta_5.$$

Definition 3.3. *Let H_3 be a free $\mathbf{Z}[\rho]$ module equipped with a skew symmetric form $\wedge$ which satisfies $\rho(v)\wedge\rho(w)=v\wedge w$. The hermitian metric h on H_3 is defined by $h(v)=v\wedge\rho(v)$. Since the value of the associated bilinear form $q(x,y)=h(x+y)-h(x)-h(y)$ on $H_3\times(1-\rho)H_3$ is divisible by 3, q mod 3 defines a $\mathbf{F}_3$ valued bilinear form q on $H_3/(1-\rho)H_3$. We denote $q(\alpha,\alpha)$ by $q(\alpha)$. Note that $q(\alpha)=2h(\alpha)$ (mod 3). (c.f.* [ATC].*)*

We apply this construction to $H_3(Y,\mathbf{Z})$. Using the basis $A_1,\dots,B_5$, one can check that the hermitian form h is isomorphic to

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}^{\oplus4}\oplus\begin{pmatrix}-2&-1\\-1&-2\end{pmatrix}$$

as a quadratic form. Moreover, $H_3(Y,\mathbf{Z})/(1-\rho)H_3(Y,\mathbf{Z})\simeq\mathbf{F}_3^5$ is equipped with a $\mathbf{F}_3$ valued quadratic form q. The multiplication by $1-\rho^2$ induces an

isomorphism from $\frac{1}{1-\rho}H_3(Y,\mathbf{Z})/H_3(Y,\mathbf{Z})$ to $H_3(Y,\mathbf{Z})\otimes\mathbf{Z}[\rho]/(1-\rho)$ and via this isomorphism q is considered as a quadratic form on

$$\frac{1}{1-\rho}H_3(Y,\mathbf{Z})/H_3(Y,\mathbf{Z})\simeq J(Y)_{1-\rho},$$

where $J(Y)_{1-\rho}$ is the $(1-\rho)$-torsion subgroup of $J(Y)$. Let $\bar{\delta}_i$ be the image of $\frac{1}{1-\rho^2}B_i$ in $\frac{1}{1-\rho}H_3(Y,\mathbf{Z})/H_3(Y,\mathbf{Z})$ under the map $\bar{c}$ in Theorem 2.6. By the compatibility of the symplectic forms on $H_1(C,\mathbf{Z})^-$ and $H_3(Y,\mathbf{Z})$ stated in Remark 2.7, the quadratic form q is

$$q(\sum_{i=1}^{5}\bar{t}_i\bar{\delta}_i)=-\sum_{i=1}^{4}\bar{t}_i^2+\bar{t}_5^2.$$

By using this explicit formula and (3.2) we can check that $\{v_1,\dots,v_5\}$ is an orthonormal basis.

We interpret the geometry of the 27 lines in a cubic surface into the finite geometry over $\mathbf{F}_3^5$ with the quadratic form q. Let $\{L_1,L_2,L_3\}$ be a tritangent of the 27 lines in X, i.e. L_1, L_2 and L_3 are contained in a hyperplane in $\mathbf{P}^3$. By choosing a marking of X, we assume that $L_1=E_1$, $L_2=L_{16}$ and $L_3=C_6$. Then we have $v_1=\Lambda([L_1]-[L_2])$. Since $[C_6]$ is fixed under the Clemens-Griffiths involution and it acts as (-1)-multiplication on the image of Λ, we have

$$\Lambda([E_1]-[C_6])=-\Lambda([L_{16}]-[C_6]). \tag{3.3}$$

Thus $\Lambda(2[C_6]-[L_{16}]-[E_1])=0$. It is easy to see that $[E_i]-[L_{i6}]$ $(i=1,\dots,5)$ and $2[C_6]-[L_{16}]-[E_1]$ generates $H_2^{prim}(X,\mathbf{Z})\otimes(\mathbf{Z}/3\mathbf{Z})$ by direct calculation. Using this basis of $H_2^{prim}(X,\mathbf{Z})\otimes(\mathbf{Z}/3\mathbf{Z})$, we can show that $q(\Lambda(v))=(v,v)$ (mod 3), where $(,)$ denotes the intersection form on $H_2^{prim}(X,\mathbf{Z})$. Therefore by the map Λ, $J(Y)_{1-\rho}$ is identified with the quotient of $H_2^{prim}(X,\mathbf{Z})\otimes(\mathbf{Z}/3\mathbf{Z})$ by the radical of the intersection form mod 3 on $H_2^{prim}(X,\mathbf{Z})$. Since the action of $W(E_6)$ on $H_2^{prim}(X,\mathbf{Z})$ preserves the intersection form, this action induces a linear map on $J(Y)_{1-\rho}$ preserving the quadratic form q. Thus we have a map $W(E_6)\to O(5,\mathbf{F}_3)$, where $O(5,\mathbf{F}_3)=\{g\in GL(5,\mathbf{F}_3)\mid g^tg=I\}$. The composite $Aut(\Gamma_{std})\to O(5,\mathbf{F}_3)\to PO(5,\mathbf{F}_3)=O(5,\mathbf{F}_3)/\{\pm I\}$ is an isomorphism.

The inverse of the above map is explicitly given as follows. By the equality (3.3), we have

$$\pm\Lambda([E_1]-[L_{16}])=\pm\Lambda([E_1]-[C_6])=\pm\Lambda([L_{16}]-[C_6]).$$

This equality implies that the map t from the set of tritangents to the subset

$$T=\{v={}^t(u_1,\dots,u_5)\in\mathbf{F}_3^5\mid q(v)=1\}/\{\pm1\}$$

of $\mathbf{F}_3^5/\{\pm1\}$ is well defined. Two elements v_1 and v_2 in T are said to be vertical if and only if $q(v_1,v_2)=0$.

Proposition 3.4. *The map*

$$t:\{\textit{ tritangents of } X\}\to T$$

is bijective. Moreover by this bijection, two tritangents are collinear, i.e. there exists a common line in them, if and only if the corresponding elements in T are vertical.

Proof. Since $\#\{$ tritangents of $X\} = \#T = 45$, it is enough to prove the injectivity of the map t. If $T_1 = \{N_1, N_2, N_3\}$ and $T_2 = \{M_1, M_2, M_3\}$ are collinear, then we showed that $t(T_1)$ and $t(T_2)$ are vertical to each other, therefore the images are different. If T_1 and T_2 are not collinear, then we may assume that $N_i \cap M_i \neq \emptyset$ for $i = 1, 2, 3$ and $N_i \cap M_j = \emptyset$ if $i \neq j$. Since two lines N_2 and M_1 are disjoint, there are exactly 5 lines which intersect both N_2 and M_1. Two of them are N_1 and M_2. The rest of them are written as F_3, E_4, E_5. Then we can find E_3 such that E_3, F_3, N_2 is a tritangent. We denote $N_1 = E_1$, $M_2 = E_2$. Then $E_1, \cdots, E_5$ are mutually disjoint lines. Moreover, there exists a unique line E_6 such that $E_1, \ldots, E_6$ are mutually disjoint lines. By blowing them down, we obtain a marking of X. In particular, $N_2 = C_6$, $M_1 = C_3$, $T_1 = \{E_1, C_6, L_{16}\}$ and $T_2 = \{E_2, C_3, L_{23}\}$. Note that the action of $\mathfrak{S}_5$ on the set of tritangents and T are equivariant. The action of the transposition (14) is trivial on $t(T_2)$ and non-trivial on $t(T_1)$, therefore $t(T_1)$ and $t(T_2)$ are different. $\square$

The explicit correspondence between the set of tritangents and the set T is given as

$$
\begin{aligned}
&t(C_6E_iL_{i6}) = v_i, \quad t(E_6C_iL_{i6}) = \sum_{j \neq i} v_j, \\
&t(E_iC_jL_{ij}) = v_j - \sum_{k \neq i,j} v_k, \quad (1 \leq i, j \leq 5), \\
&t(L_{ij}L_{kl}L_{m6}) = v_i + v_j - v_k - v_l.
\end{aligned}
$$

By this correspondence, the group $PO(5, \mathbf{F}_3)$ acts on this set T.

Since the set of the 27 lines in a cubic surface can be identified with

$$
\mathcal{L}_{cs} = \{\mu = \{T_1, \ldots, T_5\} \mid \begin{array}{l} T_i \text{ are distinct tritangents} \\ \text{which are colinear to each other.} \end{array} \},
$$

it is also identified with the set $\mathcal{L}$

$$
\mathcal{L} = \{\nu = \{T_1, \ldots, T_5\} \subset T \mid \ T_i \text{ is vertical to } T_j \text{ for } 1 \leq i < j \leq 5\}.
$$

The two lines μ and μ' corresponding to two elements ν and ν' in $\mathcal{L}$ intersect if and only if ν and ν' have a common element of T. Via this identification, $PO(5, \mathbf{F}_3)$ acts on the graph Γ_{std} and we have the map $PO(5, \mathbf{F}_3) \to Aut(\Gamma_{std})$. This map is the inverse of the map $Aut(\Gamma_{std}) \to PO(5, \mathbf{F}_3)$.

For $1 \leq i < j \leq 6$, the transposition of the index i, j for e_i, c_i, l_{ij} in Γ_{std} induces an involution of Γ_{std}, which is denoted by r_{ij}. The involutions r_{ij} generate the symmetric group $\mathfrak{S}_6$. The element r_{123} of $Aut(\Gamma_{std})$ determined

by

$$r_{123}(E_i) = L_{jk}, r_{123}(C_i) = C_i \text{ for } \{i,j,k\} = \{1,2,3\},$$
$$r_{123}(C_i) = L_{jk}, r_{123}(E_i) = E_i \text{ for } \{i,j,k\} = \{4,5,6\},$$
$$r_{123}(L_{ij}) = L_{ij} \text{ for } i \in \{1,2,3\} \text{ and } j \in \{4,5,6\}$$

is an involution. Involutions r_{ij} and r_{123} generate $Aut(\Gamma_{std})$.

On the other hand, for $1 \leq i < j \leq 5$, the linear map $R_{ij} \in PO(5, \mathbf{F}_3)$ given by

$$R_{ij}(v_k) = v_k \ (i,j \neq k), \quad R_{ij}(v_i) = v_j, \quad R_{ij}(v_j) = v_i$$

is an element of $O(5, \mathbf{F}_3)$. In general, for a vector v such that $q(v) = 2$, we define a reflection $R_v \in O(5, \mathbf{F}_3)$ with respect to v by $R_v \mid_{v\mathbf{F}_3} = -1$ and $R_v \mid_{v^\perp} = 1$. The reflections R_v for $v = (1,1,1,1,-1)$ and $(1,1,1,-1,-1)$ are denoted by R_{56} and R_{123}, respectively.

Lemma 3.5. *Under the isomorphism $\iota : Aut(\Gamma_{std}) \to PO(5, \mathbf{F}_3)$, the involution r_{ij} in $Aut(\Gamma_{std})$ corresponds to the involution R_{ij} in $PO(\mathbf{F}_3, 5)$ for $1 \leq i < j \leq 5$. Moreover the involutions r_{56} and r_{123} correspond to R_{56} and R_{123} respectively.*

Proof. The automorphism of $H_2^{prim}(X, \mathbf{Z})$ induced by R_{56} (resp. R_{123}) is the reflection with respect to the element $[E_5]-[E_6]$ (resp. $[H]-[E_1]-[E_2]-[E_3]$). By the definition of the isomorphism between $Aut(\Gamma_{std})$ and $PO(5, \mathbf{F}_3)$, r_{56} (resp. r_{123}) is the reflection with respect to $\Lambda([E_5] - [E_6])$ (resp. $\Lambda([H] - [E_1] - [E_2] - [E_3])$). Using the compatibility of q and the intersection form on $H_2^{prim}(X, \mathbf{Z})$, we get the reflection vector for r_{56} (resp. r_{123}). □

4. Zero of Theta functions restricted to curves

4.1. **Moduli space of abelian varieties with μ_3 actions.** We consider a moduli space of a certain kind of abelian varieties and its analytic expression. Let J be an abelian variety. An element of $H^2(J, \mathbf{Z})$ is called a polarization if it is the Chern class of an ample divisor. By the isomorphism $H^2(J, \mathbf{Z}) \simeq Hom(\wedge^2 H_1(J, \mathbf{Z}) \to \mathbf{Z})$, it corresponds to a symplectic form on $H_1(J, \mathbf{Z})$. For a principally polarized abelian variety J with an action of μ_3, the representation of μ_3 on $H^0(\Omega^1)$ is called the type of the action of μ_3. Let us denoted by χ the natural representation of μ_3 on $\mathbf{C}$. For a principally polarized abelian variety with an action of μ_3 of type $4\chi \oplus \bar{\chi}$, a natural $\mathbf{F}_3$-valued quadratic form q is defined on the $(1-\rho)$-torsion part $J_{1-\rho}$ of J as in Definition 3.3. A level $(1-\rho)$ structure is defined as an isomorphism $\Psi_{ab} : \oplus_{i=1}^5 \mathbf{F}_3 v_i \to J_{1-\rho}$ such that $\{\Psi_{ab}(v_1), \dots, \Psi_{ab}(v_5)\}$ is an orthonormal basis.

Definition 4.1. *The set of isomorphism classes of triples (J, ι, Ψ_{ab}) is denoted by $\mathcal{M}_{ab}$, where J is a principally polarized abelian variety, ι is a μ_3 action of type $4\chi \oplus \bar{\chi}$ and Ψ_{ab} is a level $(1-\rho)$ structure.*

Remark 4.2. *The orthogonal group $O(5, \mathbf{F}_3)$ acts as the right multiplication of $\oplus_{i=1}^5 \mathbf{F}_3 v_i$. Through the equality $g\Psi_{ab}(v) = \Psi_{ab}(vg)$, we define the left action of $O(5, \mathbf{F}_3)$ on $\mathcal{M}_{ab}$. Since (J, ι, Ψ_{ab}) is isomorphic to $(J, \iota, -\Psi_{ab})$ as triples, this action reduces to the action of $PO(5, \mathbf{F}_3)$ which is isomorphic to $W(E_6)$.*

To obtain an analytic expression of $\mathcal{M}_{ab}$, we introduce the notion of homology marking. We define the standard module Mod_{std} as the free $\mathbf{Z}$ module generated by $a_1, \dots, a_5, b_1, \dots, b_5$. On the module Mod_{std}, we define a symplectic form ϕ as $\phi(a_i, a_j) = \phi(b_i, b_j) = 0$ and $\phi(a_i, b_j) = -\delta_{ij}$ and the action of ρ as

$$(\alpha_1, \dots, \alpha_5, \beta_1, \dots, \beta_5) \begin{pmatrix} a_1 \\ \vdots \\ a_5 \\ b_1 \\ \vdots \\ b_5 \end{pmatrix} \mapsto (\alpha_1, \dots, \alpha_5, \beta_1, \dots, \beta_5)\, W \begin{pmatrix} a_1 \\ \vdots \\ a_5 \\ b_1 \\ \vdots \\ b_5 \end{pmatrix},$$

where

$$W = \begin{pmatrix} -I & -H \\ H & 0 \end{pmatrix}, \quad H = diag(1, \dots, 1, -1).$$

For a principally polarized abelian variety J with an action of μ_3 of type $4\chi \oplus \bar{\chi}$, a homology marking is defined as an isomorphism $\Psi_{hom} : Mod_{std} \to H_1(J, \mathbf{Z})$ compatible with the symplectic forms and the action of ρ. The triple (J, ι, Ψ_{hom}) is called the homology marked abelian variety with an action of μ_3 of type $4\chi \oplus \bar{\chi}$.

Definition 4.3. *The set of isomorphism classes of homology marked abelian varieties (J, ι, Ψ_{hom}) is called the moduli spaces of homology marked abelian variety and denoted by $\mathcal{M}_{hom}$.*

The opposite automorphism group $Aut^0(Mod_{std})$ is defined as the copy of $Aut(Mod_{std})$ whose product structure is defined by the reverse of that of $Aut(Mod_{std})$. The group $Aut^0(Mod_{std})$ acts on Mod_{std} from the right. The group $Aut^0(Mod_{std})$ acts on Ψ_{hom} from the left by the rule $(g\Psi_{hom})(v) = \Psi_{hom}(vg)$. If we use the basis $a_1, \dots, b_5$, Mod_{std} can be identified with the 10 dimensional integer valued row vector and $Aut^0(Mod_{std})$ is identified with the centralizer of W in $Sp(10, \mathbf{Z})$. An element g in $Aut^0(Mod_{std})$ acts on $\mathcal{M}_{hom}$ by $g(J, \iota, \Psi_{hom}) = (J, \iota, g\Psi_{hom})$.

For a homology marking $\Psi_{hom} : Mod_{std} \to H_1(J, \mathbf{Z})$, $\{A_1 = \Psi_{hom}(a_1), \dots, B_5 = \Psi_{hom}(b_5)\}$ becomes a symplectic basis. By using this basis, the module $H_1(J, \mathbf{Z})$ is identified with the integer valued row vector space $\mathbf{Z}^{10}$. Using these data, we obtain a period matrix τ in $\mathfrak{H}_5 = \{\tau \in M(5, \mathbf{C}) \mid \tau = {}^t\tau, \Im(\tau) \text{ is positive definite }\}$ such that $W \cdot \tau = \tau$ as follows. Let ϕ_i be a normalized 1-form on J. i.e. $\displaystyle\int_{B_i} \phi_j = \delta_{ij}$ and set $\tau_{ij} = \displaystyle\int_{A_i} \phi_j$. In other

words,

$$\begin{pmatrix} A_1 \\ \vdots \\ A_5 \\ B_1 \\ \vdots \\ B_5 \end{pmatrix} \begin{pmatrix} \phi_1 & , \dots , & \phi_5 \end{pmatrix} = \begin{pmatrix} \tau \\ I \end{pmatrix}.$$

Here we write $A_i \cdot \phi_j = \displaystyle\int_{A_i} \phi_j$ for simplicity. The matrix $(\tau_{ij})_{ij}$ is called the normalized period matrix. Then the normalized period matrix τ is symmetric and $\Im(\tau)$ is positive definite. Let S be the matrix for the ρ^* action on $(\phi_i)_i$, i.e.

$$\rho^*(\phi_1, \dots, \phi_5) = (\phi_1, \dots, \phi_5)S.$$

Then we have the following relation:

$$\begin{aligned} \begin{pmatrix} \tau \\ I \end{pmatrix} =& \rho_* \begin{pmatrix} A_1 \\ \vdots \\ A_5 \\ B_1 \\ \vdots \\ B_5 \end{pmatrix} (\rho^{-1})^* \begin{pmatrix} \phi_1 & , \dots , & \phi_5 \end{pmatrix} = W \begin{pmatrix} A_1 \\ \vdots \\ A_5 \\ B_1 \\ \vdots \\ B_5 \end{pmatrix} \begin{pmatrix} \phi_1 & , \dots , & \phi_5 \end{pmatrix} S^{-1} \\ =& W \begin{pmatrix} \tau \\ I \end{pmatrix} S^{-1}. \end{aligned}$$

Therefore we have $S = H\tau$ and $W \cdot \tau = \tau$, i.e. $(H\tau)^2 + H\tau + I = 0$. Here an element

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in $Sp(10, \mathbf{Z})$ acts on $\mathfrak{H}_5$ by $g \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$ for $\tau \in \mathfrak{H}_5$. Thus on the period lattice $L = \mathbf{Z}^5\tau \oplus \mathbf{Z}^5 \subset \mathbf{C}^5$, the action of ρ is given by the right multiplication of S. Under these notation, the action of ρ on $J = \mathbf{C}^5/(\mathbf{Z}^5\tau \oplus \mathbf{Z}^5)$ is also given by the right multiplication of S.

4.2. **Moduli space of abelian varieties as a ball quotient.** We give an isomorphism between the moduli space $\mathcal{M}_{hom}$ and 4-dimensional complex ball $\mathbf{B}_4 = \{{}^t(x_1, \dots, x_4) \mid\mid x_1 \mid^2 + \cdots + \mid x_4 \mid^2 < 1\}$.

Since the matrix $H\tau$ represents ρ^* on $H^0(J, \Omega^1)$ with respect to the normalized 1-forms, there exists a unique eigen vector $\eta ={}^t (\eta_1, \cdots, \eta_5) \in \mathbf{C}^5$ up to constant corresponding to the eigen value $\bar{\omega}$ of the left multiplication of

$H\tau$. In other words, we have $H\tau\eta = \bar{\omega}\eta$. Since $\Im(\tau) > 0$, we have

$$\begin{aligned} 0 <^t \bar{\eta}\Im(\tau)\eta &= {}^t\bar{\eta}\frac{\tau - \bar{\tau}}{2i}\eta \\ &= \frac{1}{2i}(-\omega^t\bar{\eta}H\eta + \omega^{2t}\bar{\eta}H\eta) \\ &= -\frac{\sqrt{3}}{2}({}^t\bar{\eta}H\eta). \end{aligned}$$

Thus we have $\mid \eta_1 \mid^2 + \cdots + \mid \eta_4 \mid^2 - \mid \eta_5 \mid^2 < 0$, so $x = {}^t(x_1, \ldots, x_4)$ is an element of $\mathbf{B}_4$, where $x_i = \dfrac{\eta_i}{\eta_5}$ $(i = 1, \ldots, 4)$. We can regard $\mathbf{B}_4$ as an open set of $\mathbf{P}^4 = \{ {}^t(\eta_1 : \cdots : \eta_5)\}$. Conversely we obtain the normalized period matrix τ from the vector x in $\mathbf{B}_4$. For any vector $\delta \in \mathbf{C}^5$ in the ω-eigen space for the left multiplication of $H\tau$, we have ${}^t\delta H\eta = 0$ because of the equalities

$$\omega \; {}^t\delta H\eta = {}^t\delta\tau\eta = \bar{\omega} \; {}^t\delta H\eta.$$

Therefore the matrix $H\tau$ can be characterized as

$$H\tau \mid_{\eta\mathbf{C}} = \omega \cdot I, \quad H\tau \mid_{\eta^{\perp_H}} = \bar{\omega} \cdot I,$$

where $\eta^{\perp_H}$ is the vector space $\{\delta \mid {}^t\delta H\eta = 0\}$. As a consequence, we have the isomorphism

$$\mathbf{B}_4 \simeq \{\tau \in \mathfrak{H}_5 \mid W \cdot \tau = \tau\} (\simeq \mathcal{M}_{hom}).$$

The module Mod_{std} is freely generated by $B_1, \ldots, B_5$ over $\mathbf{Z}[\rho]$. Since the action of $Aut^0(Mod_{std})$ is $\mathbf{Z}[\rho]$-linear, $Aut^0(Mod_{std})$ can be identified with a subgroup $U(4,1)$ of $GL(5, \mathbf{Z}[\rho])$ using the $\mathbf{Z}[\rho]$ basis $B_1, \ldots, B_5$. Here $U(4,1) = \{g \in GL(5, \mathbf{Z}[\rho]) \mid {}^t\bar{g}Hg = H\}$. For an element

$$g = A + B\rho \quad (A, B \in M(5, \mathbf{Z}))$$

of $U(4,1)$, the corresponding element $\iota(g)$ in $Sp(10, \mathbf{Z})$ is given by

$$\iota(g) = \begin{pmatrix} H(A-B)H & -HB \\ BH & A \end{pmatrix}. \tag{4.1}$$

We define the orthogonal group $O(4,1)$ by $U(4,1) \cap M(5, \mathbf{Z})$. Note that as in $\mathcal{M}_{ab}$, the action of $-\rho I \in U(4,1)$ on $\mathcal{M}_{hom}$ is trivial and the action of $U(4,1)$ on $\mathbf{B}_4$ factors through the action of $PU(4,1) = U(4,1)/ < -\rho I >$.

A homology marking $\Psi_{hom} : Mod_{std} \to H_1(J, \mathbf{Z})$, induces an isomorphism

$$\frac{1}{1-\rho} Mod_{std}/Mod_{std} \simeq J_{1-\rho}.$$

Therefore the elements $v_1, \ldots, v_5$ in $\frac{1}{1-\rho}Mod_{std}/Mod_{std}$ given by the equality (3.2) form an orthonormal basis in $\frac{1}{1-\rho}Mod_{std}$ with respect to the $\mathbf{F}_3$-valued quadratic form q. Using this basis, we get an isomorphism $\Psi_{ab} : \oplus\mathbf{F}_3 v_i \to J_{1-\rho}$ preserving the $\mathbf{F}_3$-valued quadratic form. This correspondence gives a morphism of moduli spaces $\mathcal{M}_{hom} \to \mathcal{M}_{ab}$. We define the congruence subgroup $\Gamma(1-\rho)$ by

$$\Gamma(1-\rho) = \{g \in U(4,1) \subset GL(5, \mathbf{Z}[\rho]) \mid g \equiv I \pmod{(1-\rho)}\}.$$

By the identification $\mathcal{M}_{hom} \simeq \mathbf{B}_4$, $\mathcal{M}_{ab}$ is identified with the quotient of $\mathbf{B}_4$ by the congruence subgroup $\Gamma(1-\rho)$ of $U(4,1)$. The action of $A+B\rho \in U(4,1)$ on $\mathbf{B}_4 \subset \mathbf{P}^4$ is given by ${}^t(\eta_1, \ldots, \eta_5) \mapsto (A + B\omega^2)\ {}^t(\eta_1, \ldots, \eta_5)$.

4.3. **Theta functions and their transformation formulae.** We recall the transformation formula for the acton of $Sp(10, \mathbf{Z})$ on theta constants (see [I]).

Definition 4.4. *For $(m', m'') \in \mathbf{R}^{10}$ and $\tau \in \mathfrak{H}_5$, we define theta functions $\Theta_m(\tau, z)$, $\Theta(\tau, z)$ and theta constants $\Theta_m(\tau)$ as*

$$\begin{aligned}\Theta_{(m',m'')}(\tau, z) &= \sum_{p\in\mathbf{Z}^5} \mathbf{e}(\frac{1}{2}(p + m')\tau^t(p + m') + (p + m')^t(z + m'')),\\ \Theta(\tau, z) &= \Theta_{(0,0)}(\tau, z),\\ \Theta_{(m',m'')}(\tau) &= \Theta_{(m',m'')}(\tau, 0).\end{aligned}$$

Proposition 4.5. *For any*

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(10, \mathbf{Z})$$

and $m = (m', m'') \in \mathbf{R}^{10}$, put

$$\begin{aligned}(\tau^{\#}, z^{\#}) =&((A\tau + B)(C\tau + D)^{-1}, z(C\tau + D)^{-1}),\\ m^{\#} =&mg^{-1} + \frac{1}{2}((C^tD)_0, (A^tB)_0).\end{aligned}$$

Then we have

$$\Theta_{m^{\#}}(\tau^{\#}, z^{\#}) = \mathbf{e}(\frac{1}{2}z(C\tau + D)^{-1}C^tz) \det(C\tau + D)^{1/2}u(g) \cdot \Theta_m(\tau, z) \tag{4.2}$$

where $u(g) \in \mathbf{C}^{\times}$ depends only on g.

Proposition 4.6. *Let $\tau \in \mathfrak{H}_5$ be the normalized period matrix of an abelian variety with an action of μ_3 of type $4\chi \oplus \bar{\chi}$. Then we have*

$$\Theta_{\frac{1}{2}\mathbf{1},\frac{1}{2}\mathbf{1}}(\tau, z(H\tau)^{-1}) = \mathbf{e}(\frac{1}{2}z\tau^{-1}z)\Theta_{\frac{1}{2}\mathbf{1},\frac{1}{2}\mathbf{1}}(\tau, z).$$

Proof. Since $H\tau$ is equal to the representation matrix of ρ^* for the normalized 1 forms $\phi_1, \ldots, \phi_5 \in H^0(J, \Omega^1)$, we have $\det(H\tau) = 1$. By applying the transformation formula in Proposition 4.5 for $g = W$ in §4.1, we have

$$\begin{aligned}\Theta_{\frac{1}{2}\mathbf{1},\frac{1}{2}\mathbf{1}}(\tau, z(H\tau)^{-1}) =&\mathbf{e}(\frac{1}{2}z\tau^{-1t}z)u(W)\Theta_{-\frac{1}{2}d(H),d(H)-\frac{1}{2}\mathbf{1}}(\tau, z)\\ =&\mathbf{e}(\frac{1}{2}z\tau^{-1t}z)u(W)\Theta_{\frac{1}{2}\mathbf{1},\frac{1}{2}\mathbf{1}}(\tau, z).\end{aligned}$$

By evaluating this equality at $\tau = diag(\omega, \cdots, \omega, -\omega^2)$, and $z = \mathbf{1}((H\tau)^{-1} - I)^{-1}$, we get $u(W) = 1$. Thus we get the proposition. $\square$

Definition 4.7. *Let us fix a period matrix $\tau \in \mathfrak{H}_5$. The zero $\{z \in \mathbf{C}^5 \mid \Theta(z,\tau)=0\}$ of Θ is invariant under the translation of the period lattice $\mathbf{Z}^5\tau \oplus \mathbf{Z}^5$. Therefore zero locus of Θ determines a divisor $D_{Prym,\Theta}$ of $\mathbf{C}^5/(\mathbf{Z}^5\tau\oplus\mathbf{Z}^5)$. It is called the theta divisor for this polarization.*

As a corollary of Proposition 4.6, we have the following proposition. The vector $\mathbf{1}\tau+\mathbf{1}$ is denoted by $\mathbf{I}$.

Proposition 4.8. *Let τ be an element in $\mathfrak{H}_5$ satisfying the condition of Proposition 4.6. Then the divisor $D_{Prym,\Theta}+\frac{1}{2}\mathbf{I}$ is stable under the action of ρ.*

4.4. **Zero of the restricted theta function.** In this subsection, we assume that X has no Eckardt points and apply the results in §4.1 4.2, 4.3 to the Prym variety $Prym(C,\sigma)$ for the double coverings of §2.3. Let C be a curve defined by the equation 3.1 with $a_i \in \mathbf{R}$ for $i=1,\dots,5$ and $0<a_1<\cdots<a_5$. We consider the symplectic basis $A_1,\dots,A_5,B_1,\dots,B_5$ for the half of the cup product of $H_1(C,\mathbf{Z})^-$ defined in §3.1. Let $\phi_1,\dots,\phi_5$ be a basis of the (-1)-eigen space of holomorphic 1-form for the action of σ. This is said to be normalized if

$$\int_{B_i}\phi_j=\delta_{ij}.$$

Then $A_1,\dots,B_5$ and $\phi_1,\dots,\phi_5$ are regarded as a symplectic basis for $H_1(Prym(C,\sigma),\mathbf{Z})$ and a normalized 1-form on $Prym(C,\sigma)$ for the symplectic base $A_1,\dots,B_5$. We define the normalized period matrix τ as in the last subsection. Then the Prym variety $Prym(C,\sigma)$ is isomorphic to the complex torus $\mathbf{C}^5/(\mathbf{Z}^5\tau\oplus\mathbf{Z}^5)$.

Let γ be a path in C connecting Q_0 and p. The integral of $\phi=(\phi_1,\dots,\phi_5)$ along this path is written by $\int_{Q_0}^x\phi$ for short. Since Q_0 is fixed under the action of σ, the path $\gamma-\sigma(\gamma)$ connects $\sigma(p)$ and p. The integral $\int_{\sigma(p)}^p\phi$ of ϕ along $\gamma-\sigma(\gamma)$ is a $\mathbf{C}^5$ valued holomorphic function on the universal covering of C. The theta function $\Theta_m(\tau,z)$ is denoted by $\Theta_m(z)$ for simplicity. For $v\in\mathbf{C}^5$, we consider a holomorphic function $\Theta_{\frac{1}{2}\mathbf{1},\frac{1}{2}\mathbf{1}}(v+\int_{\sigma(p)}^p\phi)$ on the universal covering of C. By the quasi periodicity of theta functions, the order of zero of the function $\Theta_{\frac{1}{2}\mathbf{1},\frac{1}{2}\mathbf{1}}(v+\int_{\sigma(p)}^p\phi)$ depends only on p in C.

First we investigate the order of zero of the function $\Theta_{\frac{1}{2}\mathbf{1},\frac{1}{2}\mathbf{1}}(v+\int_{\sigma(p)}^p\phi)$ at $p\in\Sigma=\{p_1,\dots,p_5,\sigma(p_1),\dots,\sigma(p_5),p_0,p_\infty\}$, where $v\in\frac{1}{1-\rho}L=\frac{1}{1-\rho}(\mathbf{Z}^5\tau\oplus\mathbf{Z}^5)$. The class of $\int_{\sigma(p)}^p\phi$ for $p\in\Sigma$ in $\frac{1}{1-\rho}L/L$ is denoted by $\jmath(p)$. Then the map $\jmath: C\to\mathbf{C}^5/L\simeq Prym(C,\sigma)\simeq J(Y)$ is equal to the map $\jmath$ defined in Proposition 2.9. Therefore $\jmath(p)\in\frac{1}{1-\rho}L/L$ for $p\in\Sigma$ and $\jmath(p_i)=v_i$ and $\jmath(\sigma(p_i))=-v_i$. Using the $\mathbf{F}_3$ valued quadratic form q introduced in Definition 3.3, the order of zero of theta function modulo 3 is give by the following proposition.

Proposition 4.9. *The order of zero of $\Theta(\frac{1}{2}\mathbf{I}+v+\int_{\sigma(p)}^p\phi)$ at $p\in\Sigma$ is equal to $q(v+\jmath(p))$ modulo 3.*

Next, we study the zeros of the pull back of the theta functions by the μ_3 equivariant maps $\jmath+v$ for $v \in Prym(C,\sigma)$. Even though the theta divisor depends on the choice of symplectic basis of $H_1(Prym(C,\sigma),\mathbf{Z})$, the cohomology class $c_1(D_{Prym,\Theta}) \in H^2(Prym(C,\sigma),\mathbf{Z}(1))$ of the theta divisor is independent of the choice of symplectic basis and it is equal to $\sum_{i=1}^{5} A_i^* \wedge B_i^*$, where $\{A_1^*,\dots,A_5^*,B_1^*,\dots,B_5^*\}$ is the dual basis of the symplectic basis. The degree of the inverse image of $D_{Prym,\Theta}$ by $\jmath$ is equal to the image of $c_1(D_{Prym,\Theta})$ under the morphism

$$H^2(Prym(C,\sigma),\mathbf{Z}(1)) \xrightarrow{\jmath^*} H^2(C,\mathbf{Z}(1)) \xrightarrow{\deg} \mathbf{Z}.$$

Using the relation

$$\begin{aligned}\jmath^*(A_i^*) =& \alpha_i^* - \alpha_i'^*,\\ \jmath^*(B_i^*) =& \beta_i^* - \beta_i'^*,\end{aligned}$$

and $\deg(jac^*(\alpha_i^*\wedge\beta_i^*)) = 1$ for $jac : C \to J(C)$ defined in §2.4 and $i = 1,\dots,5$, we have $\deg(\jmath^*(\sum_{i=1}^{5} A_i^*\wedge B_i^*)) = 10$. As a consequence, we have the following proposition.

Proposition 4.10. *Let $v \in Prym(C,\sigma)$. Suppose that the image of $\jmath + v : C \to Prym(C,\sigma)$ is not contained in the theta divisor $D_{Prym,\Theta}$. Then the degree of the inverse image of $D_{Prym,\Theta}$ by the map $\jmath + v$ is 10.*

Now we apply Proposition 4.9 and Proposition 4.10 to compute the multiplicities of zeros of $\Theta(v,p) = \Theta(\frac{1}{2}\mathbf{I} + v + \int_{\sigma(p)}^{p} \phi)$. For an element $v = \xi_1 v(p_1) + \cdots + \xi_5 v(p_5)$ the Hamming distance $dis(v)$ of v is defined by

$$dis(v) = \#\{i \mid \xi_i \neq 0\}.$$

Note that $q(v) = dis(v) \pmod 3$. If $dis(v) = 0,1,5$, then $\Theta(v,p) = \Theta(\frac{1}{2}\mathbf{I}+v+\int_{\sigma(p)}^{p} \phi)$ is identically zero. In fact, for example, if $dis(v) = 0$, then $\Theta(v,p) = 0$ for $p = p_1,\dots,p_5,\sigma(p_1),\dots,\sigma(p_5)$. Moreover if $p = p_0, p_\infty$, $\Theta(v,p) = 0$. Since the degree of the restriction of theta divisor is 10, $\Theta(v,p)$ is identically zero. For v such that $dis(v) = 2,3,4$, we have the following table.

The case $dis(v) = 2$, for example $v = v(p_1) + v(p_2)$,

point	p_1	p_2	p_3	p_4	p_5	p_0
order	2	2	0	0	0	2

point	$\sigma(p_1)$	$\sigma(p_2)$	$\sigma(p_3)$	$\sigma(p_4)$	$\sigma(p_5)$	p_∞
order	1	1	0	0	0	2.

The case $dis(v) = 3$, for example $v = v(p_1) + v(p_2) + v(p_3)$,

point	p_1	p_2	p_3	p_4	p_5	p_0
order	0	0	0	1	1	0

point	$\sigma(p_1)$	$\sigma(p_2)$	$\sigma(p_3)$	$\sigma(p_4)$	$\sigma(p_5)$	p_∞
order	2	2	2	1	1	0.

The case $dis(v) = 4$, for example $v = v(p_1) + v(p_2) + v(p_3) + v(p_4)$,

point	p_1	p_2	p_3	p_4	p_5	p_0
order	1	1	1	1	2	1

point	$\sigma(p_1)$	$\sigma(p_2)$	$\sigma(p_3)$	$\sigma(p_4)$	$\sigma(p_5)$	p_∞
order	0	0	0	0	2	1.

Since the degree of the function $\Theta(v,p)$ is 10, the function $\Theta(v,p)$ never vanishes outside of Σ for v such that $dis(v) = 2, 3, 4$.

4.5. **Rational map $\mathcal{M}_{ab} \to \mathbf{P}^{79}$ defined by the theta constants.** We define a rational map $\mathcal{M}_{ab} \to \mathbf{P}^{79}$ by using theta constants. Let us define a subset S of $\mathbf{F}_3^5$ by

$$S = \{v \in \mathbf{F}_3^5 - \{0\} \mid q(v) = 0\}.$$

The subset of S consisting of v such that $v \cdot r = 0$ (resp. $v \cdot r \neq 0$) is denoted by S_r (resp. $S_{\bar{r}}$), where $r = (1,1,1,1,1)$. Note that $\#S = 80$, $\#S_r = 20$ and $\#S_{\bar{r}} = 60$. We define the theta characteristic Θ_v indexed by $v = (v^{(1)}, \dots, v^{(5)}) \in S$ as follows. Let $\tilde{v}$ be an element of $\mathbf{Z}^5$ such that $\tilde{v} \equiv v$ (mod 3). Let

$$r_1 = (-1,-1,-1,-1,-1),\ r_2 = (-1,1,1,1,1),\ r_3 = (0,-1,1,0,0)$$
$$r_4 = (0,0,0,-1,1),\ r_5 = (0,1,1,-1,-1),$$

and define $\tilde{\beta} = (r_1 \cdot \tilde{v}, \dots, r_5 \cdot \tilde{v})$. Then we have

$$(r_1 \cdot \tilde{v})\bar{\delta}_1 + \cdots + (r_5 \cdot \tilde{v})\bar{\delta}_5 = v^{(1)}v_1 + \cdots + v^{(5)}v_5 \in \frac{1}{1-\rho}L/L,$$

where $\bar{\delta}_i = \frac{1}{1-\rho^2}B_i$.

Lemma 4.11. *Set*

$$\tilde{m} = \frac{1}{2}(\mathbf{1},\mathbf{1}) + \frac{1}{3}(-\tilde{\beta}H, \tilde{\beta}),$$

then the theta constant $\Theta_{\tilde{m}}^3$ is independent of the choice of a lifting $\tilde{v}$ of v to $\mathbf{Z}^5$.

Proof. If we replace $\tilde{v}$ by $\tilde{v} + 3h$ with $h \in \mathbf{Z}^5$, then $\tilde{\beta}$ and $\tilde{m}$ are replaced by $\tilde{\beta} + 3g$ and $\tilde{m} + (-gH, g)$ respectively, where $g = (r_1 \cdot h, \dots, r_5 \cdot h)$. By applying the quasi-periodicity of theta functions:

$$\Theta_{(m',m'')+(p,q)} = \mathbf{e}(m' \cdot {}^t q)\Theta_{(m',m'')}$$

for theta constants to $m' = \frac{1}{2}\mathbf{1} - \frac{1}{3}\beta H$, $m'' = \frac{1}{2}\mathbf{1} + \frac{1}{3}\beta$, $p = -gH$ and $q = g$, we have

$$\begin{aligned}
\Theta^3_{(m',m'')+(p,q)} &= \mathbf{e}(3 \cdot m' \cdot g)\Theta^3_{(m',m'')} \\
&= \mathbf{e}(3 \cdot (\frac{1}{2}(\mathbf{1},\mathbf{1}) - \frac{1}{3}\beta H) \cdot^t g)\Theta^3_{(m',m'')} \\
&= \mathbf{e}(\frac{3}{2}(\sum_{i=1}^{5} r_i \cdot h) - \beta H \cdot^t g)\Theta^3_{(m',m'')}.
\end{aligned}$$

Since the entries of $\sum_{i=1}^5 r_i$ are even, $\Theta^3_{\tilde{m}}$ is independent of the choice of $\tilde{v}$. $\square$

The theta constant $\Theta^3_{\tilde{m}}$ is denoted by Θ^3_v. Note that Θ_v is well defined only up to the multiplication of μ_3. The isomorphism $\oplus_{i=1}^5 \bar{\delta}_i \simeq \oplus_{i=1}^5 v_i$ defined by the equality (3.2) induces a morphism $\oplus_{i=1}^5 \frac{1}{1-\rho}\mathbf{Z}[\rho]B_i \to \oplus_{i=1}^5 \mathbf{F}_3 v_i$ and a homomorphism $\pi : U(4,1) \to O(5,\mathbf{F}_3)$. We have the following proposition.

Proposition 4.12. 1. *For $v \in S$, we have $\Theta^3_{-v} = -\Theta^3_v$.*

2. *The quotient $\Theta^3_v(\tau)/\Theta^3_w(\tau)$ comes to be a rational function on $\mathcal{M}_{ab} = \Gamma(1-\rho)\backslash\mathbf{B}_4$.*
3. *For $v \in S$ and $g \in O(4,1) \subset U(4,1)$ such that $(\mathbf{1}\cdot g - \mathbf{1})^t\mathbf{1} \equiv 0$ (mod 4), we have $\Theta^3_v(\iota(g)(\tau)) = c(g)\Theta^3_{v\pi(g)}(\tau)$. Here $c(g)$ is a constant depending only on $g \in O(4,1)$. As a consequence we have*

$$\frac{\Theta^3_v}{\Theta^3_w}(\iota(g)(\tau)) = \frac{\Theta^3_{v\pi(g)}}{\Theta^3_{w\pi(g)}}(\tau) \tag{4.3}$$

for $v, w \in S$.

Remark 4.13. *For any $v \in \mathbf{F}_3^5$ such that $q(v) = 2$, we can choose a lifting $\tilde{v}$ in $\frac{1}{1-\rho^2}\oplus_{i=1}^5 B_i\mathbf{Z}$ such that $h((1-\rho^2)\tilde{v}) = -2$. For example, we can choose $\frac{1}{1-\rho^2}(B_1+B_2-2B_5)$ and $\frac{1}{1-\rho^2}(4B_1+3B_2+3B_3+6B_5)$ as liftings of $\bar{\delta}_1+\bar{\delta}_2+\bar{\delta}_5$ and $\bar{\delta}_1$, respectively. The reflection for the root $(1-\rho^2)\tilde{v}$ satisfies the condition of 3 of Proposition 4.12. Therefore for any $g \in PO(5,\mathbf{F}_3)$, there exists an element $\tilde{g} \in O(4,1)$ such that the equation (4.3) holds.*

Proof. 1. We choose $-\tilde{v}$ as a lifting of $-v$. For a vector $m \in \mathbf{Q}^{10}$, we have $\Theta_{-m} = \Theta_m$ for theta constants. Therefore we have

$$\begin{aligned}\Theta_{-v} &= \Theta_{\frac{1}{2}(\mathbf{1},\mathbf{1})+\frac{1}{3}(\tilde{\beta}H,-\tilde{\beta})} \\ &= \Theta_{-\frac{1}{2}(\mathbf{1},\mathbf{1})+\frac{1}{3}(-\tilde{\beta}H,\tilde{\beta})} \\ &= \Theta_{\frac{1}{2}(\mathbf{1},\mathbf{1})+\frac{1}{3}(-\tilde{\beta}H,\tilde{\beta})+(\mathbf{1},\mathbf{1})}.\end{aligned}$$

We use the quasi-periodicity for theta constants. The equality

$$\mathbf{e}(3\cdot(\frac{1}{2}\mathbf{1} - \frac{1}{3}\beta H)\cdot^t \mathbf{1}) = -1,$$

yields $\Theta^3_{-v} = -\Theta^3_v$.

2. It is enough to prove that $\Theta_v(\tau)^3/\Theta_w(\tau)^3$ is invariant under the action of generators of $\Gamma(1-\rho)$. The group $\Gamma(1-\rho)$ is generated by complex reflections, (c.f.[ATC]) and the invariance under reflections can be proved similarly as in [Ma], Proposition 5.4, [Shi], Lemma 4.1.

3. We apply the transformation formula to compute $\Theta_m(\iota(g)\tau)$, where $m = \frac{1}{2}(\mathbf{1},\mathbf{1}) + \frac{1}{3}(-\tilde{\beta}H,\tilde{\beta})$. We use the notations $m^\#, z^\#$ of Proposition 4.5 By the definition of the embedding $O(4,1) \to Sp(10,\mathbf{Z})$ given in (4.1), we have

$$m^\# = (\frac{1}{2}(\mathbf{1},\mathbf{1}) + \frac{1}{3}(-\tilde{\beta}H,\tilde{\beta}))\iota(g) = \frac{1}{2}(\mathbf{1}HgH,\mathbf{1}g) + \frac{1}{3}(-\tilde{\beta}gH,\tilde{\beta}g).$$

Since we can choose $\tilde{v}\cdot g$ as a lifting of $v\cdot\pi(g)$, $\Theta_{v\pi(g)}$ is equal to Θ_{m_1}, where

$$m_1=\frac{1}{2}(\mathbf{1},\mathbf{1})+\frac{1}{3}(-\tilde{\beta}gH,\tilde{\beta}g).$$

To use the formula (4.2), we write $m_1=(m',m'')$, $m^{\#}=m_1+(p,q)$. Then we have

$$m'\cdot\ {}^tq=(\frac{1}{2}\mathbf{1}+\frac{-1}{3}\tilde{\beta}gH)\cdot\ {}^t(\frac{1}{2}\mathbf{1}g-\frac{1}{2}\mathbf{1}).$$

Since $\mathbf{1}g\equiv\mathbf{1}\ (\bmod 2)$, $3\cdot m'^tq$ is an integer if $(\mathbf{1}g-\mathbf{1})^t\mathbf{1}\equiv 0\ (\bmod 4)$. □

By the second part of Proposition 4.12, $\Theta=(\Theta_v)$ defines a rational map

$$\Theta=(\Theta_v)_{v\in S}:\mathcal{M}_{ab}\to\mathbf{P}^{79}. \tag{4.4}$$

5. 80 polynomials and theta constants

5.1. **Moduli space of cubic surfaces and the action of $W(E_6)$.** We introduce two moduli spaces: the moduli space $\mathcal{M}_{cs}$ of cubic surface and the moduli space $\mathcal{M}_{6pts}$ of ordered six points on the projective plane.

Definition 5.1. *The set $\{(X,\Psi_{cs})\}$ of marked cubic surface is denoted by $\mathcal{M}_{cs}$. It is equipped with a structure of algebraic variety.*

Remark 5.2. *An element g of the automorphism group $Aut(\Gamma_{std})$ of the standard graph Γ acts on $\mathcal{M}_{cs}$ from the left by $(X,\Psi_{cs})\mapsto(X,g\circ\Psi_{cs})$.*

Definition 5.3. *We define an open subset $\mathcal{M}_{6pts}$ of the moduli space of ordered 6 points in $\mathbf{P}^2$ as*

$$\mathcal{M}_{6pts}=\{p=(P_1,\dots,P_6)\in((\mathbf{P}^2)^6/Aut(\mathbf{P}^2))\mid \begin{array}{l}\textit{any 3 points are not}\\ \textit{collinear and 6 points}\\ \textit{are not coconic.}\end{array}\},$$

where the group $Aut(\mathbf{P}^2)$ acts diagonally on $(\mathbf{P}^2)^6$.

For an element (X,Ψ_{cs}), the contraction Z of X by the (-1)-lines corresponding to $e_1,\dots,e_6$ is known to be isomorphic to $\mathbf{P}^2$. Therefore the images $P_1,\dots,P_6$ of $e_1,\dots,e_6$ in Z gives an element of $\mathcal{M}_{6pts}$. This correspondence defines a morphism form $\mathcal{M}_{cs}$ to $\mathcal{M}_{6pts}$, which is known to be isomorphic.

The group $Aut(\Gamma_{std})$ acts on $\mathcal{M}_{6pts}$ via this isomorphism. Here we describe the action of $Aut(\Gamma_{std})$ on $\mathcal{M}_{6pts}$ for the later use. The involution r_{ij} maps $(P_1,\dots,P_i,\cdots,P_j,\dots,P_6)\mapsto(P_1,\dots,P_j,\cdots,P_i,\dots,P_6)$. Let $b_1:\tilde{P}\to\mathbf{P}^2$ be the blowing up of $\mathbf{P}^2$ with the center $P_1\cup P_2\cup P_3$ and $b_2:\tilde{P}\to\mathbf{P}^2$ be the contraction of the strict transforms of L_{23},L_{31} and L_{12} to points P_1',P_2' and P_3'. The composite rational map $b_2\circ b_1^{-1}$ is denoted by Q_{123}. The involution r_{123} corresponding to the map

$$(P_1,\dots,P_6)\mapsto(P_1',P_2',P_3',Q_{123}(P_4),Q_{123}(P_5),Q_{123}(P_6))$$

can be described as follows. If four points $P_1,\dots,P_4$ are generic position, using the action of $GL(3,\mathbf{C})$, we can normalize them as $(1:0:0),(0:1:$

$0), (0:0:1)$, and $(1:1:1)$. If we write the points P_5, P_6 as $(1:x_1:x_3), (1:x_2:x_4)$, then the image of this $(P_1,\dots,P_6)$ under r_{123} is

$$((1:0:0),(0:1:0),(0:0:1),(1,1,1),(1,x_1^{-1},x_3^{-1}),(1,x_2^{-1},x_4^{-1})).$$

We define a morphism *per* from $\mathcal{M}_{cs}$ to $\mathcal{M}_{ab}$ after Allcock-Carlson-Toledo. Let $(X,\Psi_{cs}:\Gamma(X)\to\Gamma_{std})$ be a marked cubic surface. The line $\Psi_{cs}^{-1}(e_i)$, $\Psi_{cs}^{-1}(c_i)$ and $\Psi_{cs}^{-1}(l_{ij})$ are denoted as E_i, C_i and L_{ij} respectively. Let Y be the μ_3-covering of $\mathbf{P}^3$ branching along X. Then $v_i=\Lambda([E_i]-[L_{i6}])$ $(i=1,\dots,5)$ forms a orthonormal basis of $J(Y)_{1-\rho}$ and this basis defines a level $(1-\rho)$-structure $\Psi_{ab}:\mathbf{F}_3^5\to J(Y)$. With the natural polarization on $J(Y)$, $(J(Y),\iota,\Psi_{ab})$ is an element of $\mathcal{M}_{ab}$.

Definition 5.4. *The above correspondence*

$$(X,\Psi_{cs})\mapsto(J(Y),\iota,\Psi_{ab})$$

defines a morphism from $\mathcal{M}_{cs}$ to $\mathcal{M}_{ab}$. This morphism is called the period map for cubic surfaces and denoted by per.

Proposition 5.5. *The actions of $Aut(\Gamma_{std})$ and $PO(\mathbf{F}_3,5)$ on $\mathcal{M}_{cs}$ and $\mathcal{M}_{ab}$ are equivariant via the morphism per.*

5.2. **Projective embedding of $\mathcal{M}_{cs}$ defined by 80 polynomials.** Let $\tilde{P}_i={}^t(P_{i1},P_{i2},P_{i3})$ be an element in $\mathbf{C}^3$. For each element $v\in S$, we attach relatively invariant polynomials Z_v on $(\mathbf{C}^3)^6$ under the action of $(g,t)\in GL(3,\mathbf{C})\times(\mathbf{C}^\times)^6$ defined by $\tilde{p}=(\tilde{P}_1,\dots,\tilde{P}_6)\mapsto(t_1g\tilde{P}_1,\cdots,t_6g\tilde{P}_6)$, where $t=(t_1,\dots,t_6)$.

The determinant $\det(\tilde{P}_i,\tilde{P}_j,\tilde{P}_k)$ is denoted by D_{ijk}. We define

$$\tilde{P}_i^{(2)}={}^t(P_{i1}^2,P_{i2}^2,P_{i3}^2,P_{i1}P_{i2},P_{i2}P_{i3},P_{i3}P_{i1})$$

and $D_{1\cdots6}=\det(\tilde{P}_1^{(2)},\dots,\tilde{P}_6^{(2)})$. Let i,j,k,l,m be distinct elements in $\{1,2,3,4,5\}$. For $v_i+v_j+v_k\in S$, we define a polynomial $Z_{v_i+v_j+v_k}$ by

$$(-1)^{i+j+k}Z_{v_i+v_j+v_k}=D_{ijk}D_{lm6}D_{1\cdots6},$$

if $i<j<k$, $l<m$. In the same way, we define a polynomial $Z_{v_k+v_l-v_m}$ by

$$Z_{v_k+v_l-v_m}=D_{ikl}D_{jkl}D_{km6}D_{lm6}D_{mij}D_{6ij}.$$

For general $v\in S$, we define Z_v by the rule $Z_{-v}=-Z_v$. If $\tilde{P}_i$ is not zero, $\tilde{P}_i$ determines a point in $\mathbf{P}^2$ and it is denoted by P_i. Note that Z_v's are not zero if $p=\{P_1,\dots,P_6\}\in(\mathbf{P}^2)^6$ is a point of $\mathcal{M}_{6pts}$. Therefore $(Z_v)_{v\in S}$ defines a morphism to the projective space $\mathbf{P}^{79}$:

$$Z=(Z_v)_{v\in S}:\mathcal{M}_{6pts}\to\mathbf{P}^{79}. \tag{5.1}$$

According to [C], [DO] and [Y], this projective morphism is actually an embedding. Via the isomorphism from $\mathcal{M}_{cs}$ to $\mathcal{M}_{6pts}$, we regard the morphism Z as that from $\mathcal{M}_{cs}$ to $\mathbf{P}^{79}$. The subgroup in $O(\mathbf{F}_3,5)$ generated by R_{ij} $(i<j)$ is isomorphic to $\mathfrak{S}_6$, and R_{ij} corresponds to the transposition (ij) via this isomorphism.

and it is identified with $\mathfrak{S}_6$. An element g in $\mathfrak{S}_6$ acts on $(\mathbf{C}^3)^6$ by

$$\tilde{P} = (\tilde{P}_1, \dots, \tilde{P}_6) \mapsto g \cdot \tilde{P} = (\tilde{P}_{\sigma^{-1}(1)}, \dots, \tilde{P}_{\sigma^{-1}(6)}).$$

Note that this action is compatible with the action of $Aut(\Gamma_{std})$ on $\mathcal{M}_{6pts}$.

Proposition 5.6. *For an element $g \in \mathfrak{S}_6$, we have*

$$Z_v(g \cdot \tilde{P}) = Z_{vg}(\tilde{P}).$$

Proof. It is enough to prove the proposition for $R_{12}, \dots, R_{56} \in \mathfrak{S}_6$. Note that if $v \in S_r$ (resp. $v \in S_{\bar{r}}$) then $v \cdot g \in S_r$ (resp. $v \in S_{\bar{r}}$). Therefore we can check that the proposition by the definition of Z_v. □

For the statement of the main theorem, we recall notations.

- $\mathcal{M}_{cs}$: The moduli space of the marked cubic surfaces. See Definition 5.1.
- $\mathcal{M}_{ab}$: The moduli space of 5-dimensional abelian varieties with actions of μ_3 of type $4\chi \oplus \bar{\chi}$ and level $(1-\rho)$-structures. See Definition 4.1.
- Z: The morphism from $\mathcal{M}_{cs}$ to $\mathbf{P}^{79}$ defined by 80 polynomial after Coble. See (5.1) in §5.2.
- *per*: The period map for the cubic surfaces after Allcock-Carlson-Toledo. See Definition 5.4.
- Θ: The morphism form $\mathcal{M}_{ab}$ to $\mathbf{P}^{79}$ defined by 80 theta constants. See (4.4) in §4.5.

The main theorem of this paper is the following.

Theorem 5.7 (Main Theorem)**.** *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{M}_{cs} \arrow[dr, "Z"] \arrow[dd, "per"] & \\
& \mathbf{P}^{79} \\
\mathcal{M}_{ab} \arrow[ur, "\Theta"'] &
\end{tikzcd}$$

In other words, $(Z_v)_{v \in S} = (\Theta_v)_{v \in S}$ in $\mathbf{P}^{79}$. In particular, per is an isomorphism from $\mathcal{M}_{cs}$ to the open set U of $\mathcal{M}_{ab}$ defined by $\cap_{v \in S}\{\Theta_v \neq 0\}$.

Note that the second statement is announced in [ATC].

5.3. **Proof of Theorem 5.7.**

Proposition 5.8. *Via the morphism per : $\mathcal{M}_{6pts} \to \mathcal{M}_{ab}$, we have*

$$\frac{\Theta^3_{v_1+v_2+v_3}}{\Theta^3_{v_1+v_2-v_3}} = c \cdot \frac{Z_{v_1+v_2+v_3}}{Z_{v_1+v_2-v_3}}, \tag{5.2}$$

where c is a 6-th root of unity.

Proof. In this proof, for two non zero functions f, g we write $f \approx g$ if there exists a 6-th root of unity c such that $f = c \cdot g$. To prove the equality (5.2), it enough to show this on the open set of M_{cs} corresponding to cubic surfaces with no Eckardt points. We use the normal form of 6 points as in §2.1;

$$P_i = (1 : a_i^2 : a_i) \text{ for } i = 1, \dots, 5, \quad P_6 = (0 : 0 : 1).$$

Since $D_{ijk} \approx (a_i - a_j)(a_j - a_k)(a_k - a_i)$ for $i, j, k \neq 6$, $D_{ij6} \approx a_i^2 - a_j^2$ and $D_{1\cdots 6} \approx \prod_{1 \leq i < j \leq 5}(a_i - a_j)$, we have

$$\frac{Z_{v_1+v_2+v_3}}{Z_{v_1+v_2-v_3}} \approx \frac{(a_3 - a_2)(a_3 - a_1)}{(a_3 + a_2)(a_3 + a_1)}. \tag{5.3}$$

For a point $p = (x, y) \in C$, we define a $\mathbf{C}^5$ multi-valued function $\iota(p) = \int_{\sigma(p)}^{p} \phi$. As we mentioned, $\iota(p)$ depends on the path connecting p_0 and p. We choose a lifting $\tilde{v}_i = \frac{1}{3}(-s_i H\tau + s_i)$ of v_i in $\frac{1}{1-\rho}L$. Then $\rho(\tilde{v}_i) = \frac{1}{3}(2s_i H\tau + s_i)$ and $\rho^2(\tilde{v}_i) = \frac{1}{3}(-s_i H\tau - 2s_i)$. By the quasi-periodicity for theta functions, the multivalued meromorphic function

$$f(p) = \frac{\prod_{i=0}^{2} \Theta(\frac{1}{2}\mathbf{I} + \rho^i(\tilde{v}_2 + \tilde{v}_3) + \iota(p))}{\prod_{i=0}^{2} \Theta(\frac{1}{2}\mathbf{I} + \rho^i(\tilde{v}_2 - \tilde{v}_3) + \iota(p))}, \tag{5.4}$$

on p becomes a single valued rational function on C. Using the table of §4.4, we have the following table on the order of zero of the numerator and the denominator at $p_i = (a_i, 0)$ of (5.4):

point	p_1	p_2	p_3	p_4	p_5	p_0
order of numerator	0	6	6	0	0	6
order of denominator	0	6	3	0	0	6

point	$\sigma(p_1)$	$\sigma(p_2)$	$\sigma(p_3)$	$\sigma(p_4)$	$\sigma(p_5)$	p_∞
order of numerator	0	3	3	0	0	6
order of denominator	0	3	6	0	0	6

As a consequence, the rational function $f(p)$ is equal to $c \cdot \dfrac{x - a_3}{x + a_3}$, where c is a constant independent of p.

First we evaluate the function $f(p)$ at p_1. The value $f(p_1) = c \cdot \dfrac{a_1 - a_3}{a_1 + a_3}$ is equal to

$$\begin{aligned} c \cdot \frac{a_1 - a_3}{a_1 + a_3} &= \frac{\prod_{i=0}^{2} \Theta(\frac{1}{2}\mathbf{I} + \rho^i(\tilde{v}_2 + \tilde{v}_3) + \tilde{v}_1)}{\prod_{i=0}^{2} \Theta(\frac{1}{2}\mathbf{I} + \rho^i(\tilde{v}_2 - \tilde{v}_3) + \tilde{v}_1)} \\ &\approx \frac{\prod_{i=0}^{2} \Theta_{\frac{1}{2}(\mathbf{1},\mathbf{1}) + \rho^i(\tilde{v}_2 + \tilde{v}_3) + \tilde{v}_1}}{\prod_{i=0}^{2} \Theta_{\frac{1}{2}(\mathbf{1},\mathbf{1}) + \rho^i(\tilde{v}_2 - \tilde{v}_3) + \tilde{v}_1}} \cdot \mathbf{e}(\frac{4}{3} s_2 H\tau H^t s_3). \end{aligned} \tag{5.5}$$

Here we used the relation between theta functions and theta constants:

$$\Theta_{(a,b)} = \Theta(a\tau + b)\mathbf{e}(\frac{1}{2} a\tau^t a + a^t b).$$

Next we consider the limit $\lim_{x\to\sigma(p_2)} f(p) = c\cdot\dfrac{-a_2-a_3}{-a_2+a_3}$. To compute the limit of theta functions, we choose a path from Q_0 to p such that $\iota(p)$ tends to $-\tilde{v}_2$ if p tends to $\sigma(p_2)$. Using this path, $h=\tilde{v}_2+\iota(p)$ is a function on C defined on a neighborhood of $\sigma(p_2)$. We can choose a local parameter u of C at $\sigma(a_2)$ such that $u(\sigma(p_2))=0$ and $h(-u)=-h(u)$. Since the order of zero of

$$F(u)=\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)+\iota(p))=\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)-\tilde{v}_2+h(u))$$

at $u=0$ is one, we have $\displaystyle\lim_{u\to 0}\frac{F(u)}{F(-u)}=-1$. Therefore we have

$$\begin{aligned}(5.6)\qquad &\lim_{x\to\sigma(p_2)}\frac{\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)+\iota(p))}{\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2-\tilde{v}_3)+\iota(p))}\\ &=\lim_{u\to 0}\frac{\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)-\tilde{v}_2+h(u))}{\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2-\tilde{v}_3)-\tilde{v}_2+h(u))}\\ &=\lim_{u\to 0}\frac{\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)-\tilde{v}_2+h(u))}{\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)-\tilde{v}_2-h(u)+w)},\end{aligned}$$

where $w=-(\mathbf{1},\mathbf{1})+2\tilde{v}_2-2\rho^i(\tilde{v}_2)\in\mathbf{Z}^{10}$. Put $w=(w',w'')$ and use the equality

$$\Theta(z+a\tau+b)=\mathbf{e}(-\frac{1}{2}a\tau{}^t a-a{}^t z)\Theta(z),$$

for $a,b\in\mathbf{Z}^5$, then we have

$$\begin{aligned}&\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)-\tilde{v}_2-h(u)+w)\\ =&\Theta(\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)-\tilde{v}_2-h(u))\cdot\mathbf{e}(-\frac{1}{2}w'\tau{}^t w'-w'{}^t z_i),\end{aligned}$$

where $z_i=\frac{1}{2}\mathbf{I}+\rho^i(\tilde{v}_2+\tilde{v}_3)-\tilde{v}_2-h(u)$. Therefore the limit (5.6) is equal to

$$-\mathbf{e}(\frac{1}{2}w'\tau{}^t w'+w'{}^t(\frac{1}{2}\mathbf{1}+\rho^i(\tilde{v}_2+\tilde{v}_3)-\tilde{v}_2)).$$

Multiplying the equality (5.6) for $i=0,1,2$, we have

$$(5.7)\qquad c\cdot\frac{-a_2-a_3}{-a_2+a_3}\approx\mathbf{e}(\frac{4}{3}s_2H\tau H^t s_3).$$

By the equality (5.5) and (5.7), $\Theta^3_{v_1+v_2+v_3}/\Theta^3_{v_1+v_2-v_3}$ is equal to the right hand side of (5.3).

□

The following combinatorial lemma is straight forward.

Lemma 5.9. *We put* $e_0=v_1+v_2-v_3, f_0=v_3+v_5-v_4$ *(resp.* $e_0'=v_1+v_2+v_6, f_0'=v_1+v_2+v_3$*). Then for any elements* $e,f\in S_{\bar{r}}$ *(resp.* $e',f'\in S_r$*), there exist* $g_1,\dots,g_k\in\mathfrak{S}_6$ *(resp.* $g_1',\dots,g_k'\in\mathfrak{S}_6$*) such that* $e=e_0g_1, f_0g_1=e_0g_2,\dots,f_0g_k=f$ *(resp.* $e'=e_0'g_1', f_0'g_1'=e_0'g_2',\dots,f_0'g_k'=f'$*).*

For elements $v, w \in S$, we consider the following equality $EC(v, w, c)$:

$$EC(v,w,c) \qquad \frac{\Theta_v^3}{\Theta_w^3} = c \cdot \frac{Z_v}{Z_w}.$$

For example by Proposition 5.8, the equality $EC(v_1+v_2+v_3, v_1+v_2-v_3, c)$ holds. By Proposition 4.12 and Proposition 5.6, the equality $E(v, w, c)$ implies $E(vg, wg, c)$ for $g \in \mathfrak{S}_6$.

Lemma 5.10. *For elements $v, w \in S_r$, (resp. $v, w \in S_{\bar{r}}$) the statement $EC(v, w, 1)$ holds.*

Proof. By applying $(56) \in \mathfrak{S}_6$ to the equality $EC(v_1+v_2+v_3, v_1+v_2-v_3, c)$, we have the equality $EC(v_1+v_2+v_3, v_3+v_5-v_4, c)$. Therefore we have

$$\begin{aligned}\frac{Z_{v_1+v_2-v_3}}{Z_{v_3+v_5-v_4}} &= c \cdot \frac{Z_{v_1+v_2+v_3}}{Z_{v_3+v_5-v_4}} \cdot c^{-1} \cdot \frac{Z_{v_1+v_2-v_3}}{Z_{v_1+v_2+v_3}} \\ &= \frac{\Theta^3_{v_1+v_2+v_3}}{\Theta^3_{v_3+v_5-v_4}} \cdot \frac{\Theta^3_{v_1+v_2-v_3}}{\Theta^3_{v_1+v_2+v_3}} \\ &= \frac{\Theta^3_{v_1+v_2-v_3}}{\Theta^3_{v_3+v_5-v_4}}.\end{aligned}$$

Thus we have the equality $EC(e_0, f_0, 1)$, where $e_0 = v_1 + v_2 - v_3$, $f_0 = v_3 + v_5 - v_4$. For given $e, f \in S_{\bar{r}}$, we can choose $g_1, \dots, g_k \in \mathfrak{S}_6$ such that $e = e_0 g_1, f_0 g_1 = e_0 g_2, \dots, v_f g_k = f$ by Lemma 5.9. Using these elements $g_1, \dots, g_k$, we have

$$\frac{Z_e}{Z_f} = \frac{Z_{e_0 g_1}}{Z_{f_0 g_1}} \frac{Z_{e_0 g_2}}{Z_{f_0 g_2}} \cdots \frac{Z_{e_0 g_k}}{Z_{f_0 g_k}},$$

and the similar equality for $\dfrac{\Theta_e^3}{\Theta_f^3}$. The equality $E(e_0, f_0, 1)$ implies $E(e_0 \cdot g_i, f_0 \cdot g_i, 1)$ and as a consequence, we have the equality $E(e, f, 1)$ for $e, f \in S_{\bar{r}}$.

In the same way by applying $(36) \in \mathfrak{S}_6$ to the equality $EC(v_1+v_2+v_3, v_1+v_2-v_3, c)$, we get the equality $EC(v_1+v_2+v_6, v_1+v_2-v_3, c)$. By taking quotient, we have $EC(e_0, f_0, 1)$, where $e_0 = v_1+v_2+v_6$, $f_0 = v_1+v_2+v_3$. Again using Lemma 5.9 and the same technic as $S_{\bar{r}}$, we get $EC(e, f, 1)$ for $e, f \in S_r$. □

Proof of the Main Theorem. We can directly check the equality

$$\frac{Z_{v_3+v_4-v_5}}{Z_{v_1+v_2-v_5}}(r_{123}(x)) = \frac{Z_{-v_1-v_2-v_4}}{Z_{v_2+v_4-v_5}}(x).$$

By Proposition 4.12, we have the equality $EC(-v_1-v_2-v_4, v_2+v_4-v_5, 1)$. As a consequence we have $EC(v, w, 1)$ for all $v, w \in S$ and get the commutativity of the diagram in the main theorem.

We prove the last part of the theorem. By [Y], the map $\mathcal{M}_{cs} \to U$ is an isomorphism. Thus the map *per* is proper morphism and the image contains an open dense subset and it is surjective. By the commutativity of diagram, since *per* is injective, it is isomorphism. Thus we get the last statement of the main theorem. □

5.4. **Relations for theta constants.** As an application of the main theorem §5.7, we prove identities satisfied by theta constants. The following linear relation between Z_v's is one of the Plücker relation for (3×3)-minors in (3×6)-matrices :

$$Z_{v_1} + Z_{v_2} + Z_{v_3} + Z_{v_4} = 0,$$

where $v_1 = (1,1,0,1,0)$, $v_2 = -(1,0,0,1,1)$, $v_3 = (0,1,1,0,1)$ and $v_4 = -(1,0,1,1,0)$. Since the elements $\pm v_1, \dots, \pm v_4$ are characterized as the set of $S/\{\pm 1\}$ vertical to $w_1 = (1,1,1,1,1)$ and $w_2 = (1,0,0,-1,0)$. Note that $q(w_1) = q(w_2) = 2$, $w_1 \cdot w_2 = 0$ and the set $\{v_1, \dots, v_4\}$ satisfies the condition $v_i \cdot v_j = 1$ if $i \neq j$. If another representative $\{v_1', \dots, v_4'\}$ of $\pm v_1, \dots, \pm v_4$ satisfies the same condition, then $v_1' = v_1, \dots, v_4' = v_4$ or $v_1' = -v_1, \dots, v_4' = -v_4$. Since on the set

$$R = \{ \text{ unordered pair } (w_1, w_2) \in (\mathbf{P}(\mathbf{F}_3^5))^2 \mid q(w_1) = q(w_2) = 2, w_1 \cdot w_2 = 0\},$$

the group $PO(\mathbf{F}_3, 5)$ acts transitively, we have the following cubic relations.

Corollary 5.11. *For $\mathbf{w} = (w_1, w_2) \in R$, we define $I(\mathbf{w})$ by $I(\mathbf{w}) = \{v \in S/\{\pm 1\} \mid v \cdot w_1 = v \cdot w_2 = 0\}$. Then we have*

$$\sum_{v \in \tilde{I}(\mathbf{w})} \Theta_v^3 = 0.$$

Here we choose a representative $\tilde{I}(\mathbf{w}) = \{v_1, \dots, v_4\}$ of $I(\mathbf{w})$ such that $v_i \cdot v_j = 1$ if $i \neq j$.

On the other hand, one can prove the following cubic relation for polynomials $\{Z_v\}$ (See [Y]):

$$Z_{v_1} Z_{v_2} Z_{v_3} = Z_{w_1} Z_{w_2} Z_{w_3},$$

where

$$\begin{aligned} &v_1 = (1,1,0,1,0),\ v_2 = (0,1,1,-1,0),\ v_3 = (1,-1,1,0,0) \\ &w_1 = -(1,0,0,1,1),\ w_2 = (1,0,1,0,-1),\ w_3 = (0,0,1,-1,1). \end{aligned}$$

If we put $u = (-1,0,1,1,0)$, then the sets $V_1 = \{\pm v_1, \pm v_2, \pm v_3, \pm u, 0\}$ and $V_2 = \{\pm w_1, \pm w_2, \pm w_3, \pm u, 0\}$ are maximal totally isotropic subspaces in $\mathbf{F}_3^5$. To determine the signature of the equality we consider $\epsilon = \prod_{i,j}(v_i \cdot w_j)$. If we change one of the signatures of $v_1, \dots, w_3$, then the signature ϵ changes. We define the set

$$Q = \{ \text{ unordered pair } (V_1, V_2) \mid \begin{array}{l} V_1 \text{ and } V_2 \text{ are maximal totally isotropic} \\ \text{subspaces of } \mathbf{F}_3^5 \text{ and } V_1 \cap V_2 \text{ is one di-} \\ \text{mensional } \mathbf{F}_3 \text{ subspace.} \end{array} \}.$$

Then the action of $PO(\mathbf{F}_3, 5)$ on Q is transitive, we have the following identity of degree 9.

Corollary 5.12. *Let (V_1, V_2) be an element of Q. Choose a representative S_{V_1} and S_{V_2} of $\mathbf{P}(V_1) - \mathbf{P}(V_1 \cap V_2)$ and $\mathbf{P}(V_2) - \mathbf{P}(V_1 \cap V_2)$ in S. Then we*

have the following identity

$$\prod_{v_1 \in S_{V_1}} \Theta_{v_1}^3 = \tilde{\epsilon}(S_{V_1}, S_{V_2}) \prod_{v_2 \in S_{V_2}} \Theta_{v_2}^3.$$

Here the lifting of $\prod_{v_1 \in S_{V_1}, v_2 \in S_{V_2}} (v_1 \cdot v_2)$ *to* $\{\pm 1\}$ *is denoted by* $\tilde{\epsilon}(S_{V_1}, S_{V_2})$.

Remark 5.13. *Note that the system of equations*

$$\prod_{v_1 \in S_{V_1}} Z_{v_1} = \tilde{\epsilon}(S_{V_1}, S_{V_2}) \prod_{v_2 \in S_{V_2}} Z_{v_2} \quad ((V_1, V_2) \in Q)$$

$$\sum_{v \in \tilde{I}(\mathbf{w})} Z_v = 0 \quad (\mathbf{w} \in R)$$

is a defining system of equations of the closure of $\mathcal{M}_{6pts}$ *in* $\mathbf{P}^{79}$ *(see* [Y]*).*

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Division of Mathematics, Graduate School of Science, Hokkaido University, Japan

E-mail address: `matsu@math.sci.hokudai.ac.jp`

Department of Mathematical Science, University of Tokyo, Komaba, Meguro, Japan

E-mail address: `terasoma@ms.u-tokyo.ac.jp`